Unit D2

Sequences

Introduction

This unit deals with sequences of real numbers, such as

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \ldots,$$

$$0, 1, 0, 1, 0, 1, \ldots,$$

$$1, 2, 4, 8, 16, 32, \ldots.$$

The three dots (an ellipsis) indicate that the sequence continues indefinitely.

You will learn about various properties that a sequence may possess, the most important of which is *convergence*. Roughly speaking, a sequence is *convergent*, or *tends to a limit*, if the numbers in the sequence approach arbitrarily close to a unique real number, which is called the *limit* of the sequence. For example, you will see that the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \ldots,$$

is convergent with limit 0. On the other hand, the numbers in the sequence

$$0, 1, 0, 1, 0, 1, \ldots,$$

do not approach arbitrarily close to a unique real number, so this sequence is not convergent. Likewise, the sequence

$$1, 2, 4, 8, 16, 32, \ldots,$$

is not convergent. A sequence which is not convergent is called *divergent*.

One of the reasons for studying sequences is that they provide a relatively simple setting in which we can begin to explore precise definitions of these ideas of convergence and limits. As you will see in the next unit, sequences are also a key tool in deciding when and how infinite sums make sense.

Intuitively, it seems plausible that some sequences are convergent, whereas others are not. However, the above description of convergence, involving the phrase 'approach arbitrarily close to', lacks the precision required in pure mathematics. If we wish to work in a serious way with convergent sequences, prove results about them and decide beyond doubt whether or not a given sequence is convergent, then we need a rigorous definition of this concept.

Historically, such a definition emerged only in the late nineteenth century, when mathematicians such as Bolzano, Cantor, Cauchy, Dedekind and Weierstrass placed analysis on a rigorous footing. It is not surprising, therefore, that at first sight the definition of convergence is rather subtle and it may take you a little time to grasp it fully.

1 Introducing sequences

In this section you will see how to picture the behaviour of a sequence by drawing a *sequence diagram*. You will also study *monotonic* sequences, that is, sequences which are either increasing or decreasing.

1.1 What is a sequence?

Ever since you learned to count, you have been familiar with the sequence of natural numbers

$$1, 2, 3, 4, 5, 6, \ldots .$$

You will also have encountered many other sequences of numbers, such as

$$2, 4, 6, 8, 10, 12, \ldots ,$$

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{64}, \ldots .$$

We begin our study of sequences with some definitions and notation.

Definitions

A **sequence** is an unending list of real numbers

$$a_1, a_2, a_3, \ldots .$$

The real number a_n is called the ***n*th term** of the sequence, and the sequence is denoted by

$$(a_n).$$

In the examples above, it is assumed that all the terms of the sequence after the first few are obtained by continuing the pattern in an obvious way. However, it is usually better to give a precise description of a typical term of the sequence, and often we can do this by giving an explicit formula for the nth term. For example, the nth term of the sequence (a_n) whose first few terms are

$$1, 3, 5, 7, 9, 11, \ldots ,$$

is given by the formula

$$a_n = 2n - 1, \quad n = 1, 2, \ldots .$$

We often refer to a sequence by writing the formula for its nth term in round brackets. In this notation, the sequence (a_n) would be written as $(2n - 1)$, where it is understood that n takes the successive values $1, 2, \ldots$.

Remarks

1. Although most texts on analysis use the notation (a_n) for a sequence, you may also come across the alternative notations $\{a_n\}$ or $\langle a_n \rangle$.
2. Of course, round brackets are used frequently in mathematics with a variety of different meanings, so there is some risk of ambiguity in using the notation (a_n) for a sequence. This is especially true when we refer to a sequence by writing the formula for its nth term in round brackets, as with the sequence $(2n - 1)$ mentioned above. Usually, the context will make clear whether or not an expression in round brackets refers to a sequence. For example, if it appears in the middle of an equation, an expression in round brackets does *not* denote a sequence. Thus in the equation $2n^2 - n = n(2n - 1)$, the expression $(2n - 1)$ does not refer to a sequence.
3. Notice that a sequence of real numbers differs from a *set* of real numbers. Changing the order of the terms in a sequence gives us a new sequence, whereas rearranging the elements of a set leaves the set unchanged. Moreover, the same number can occur many times in a sequence, but not in a set; for example, all the terms in the sequence

   $$0, 1, 0, 1, 0, 1, \dots,$$

   belong to the set $\{0, 1\}$. One advantage of the round bracket notation for sequences is that it avoids any confusion with sets that might arise from using braces (curly brackets).
4. For all the sequences you will meet in this unit, there will be an explicit formula for the nth term. However, this is not essential – for example, the sequence of digits in the decimal expansion of π is a well-defined sequence, but there is no formula for its nth term.

Exercise D22

Calculate the first five terms of each of the following sequences (a_n).

(Give your answer to part (e) to two decimal places.)

(a) $a_n = 3n + 1, \quad n = 1, 2, \dots$

(b) $a_n = 3^{-n}, \quad n = 1, 2, \dots$

(c) $a_n = (-1)^n n, \quad n = 1, 2, \dots$

(d) $a_n = n!, \quad n = 1, 2, \dots$

(e) $a_n = \left(1 + \dfrac{1}{n}\right)^n, \quad n = 1, 2, \dots$

Sequences often begin with a term corresponding to $n = 1$. However, sometimes it is necessary (or convenient) to begin a sequence with some other value of n. For example, the sequence (a_n) defined by

$$a_n = 1/(n! - n)$$

cannot begin with $n = 1$ or 2. We indicate this by writing, for example, $(a_n)_3^\infty$ to represent the sequence

$$a_3, a_4, a_5, \ldots$$

or writing

$$a_n = 1/(n! - n), \quad n = 3, 4, \ldots.$$

If a sequence is written as (a_n) with no subscripts or superscripts, then we assume that this denotes the sequence $(a_n)_1^\infty$.

Sequence diagrams

One helpful way to think of a sequence is as a *function* f with domain $\mathbb{N}$ and codomain $\mathbb{R}$ that maps each number n in the domain to the nth term of the sequence:

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{R} \\ n &\longmapsto a_n \end{aligned}$$

Using this idea, we can picture how a given sequence (a_n) behaves by drawing its **sequence diagram**, that is, the graph of the function from $\mathbb{N}$ to $\mathbb{R}$ that represents the sequence. To do this, we mark suitable values of n on the horizontal axis and, for each value of n, we plot the point (n, a_n). Often it is necessary to use different scales on the axes for clarity. Figure 1 shows the sequence diagrams for three different sequences.

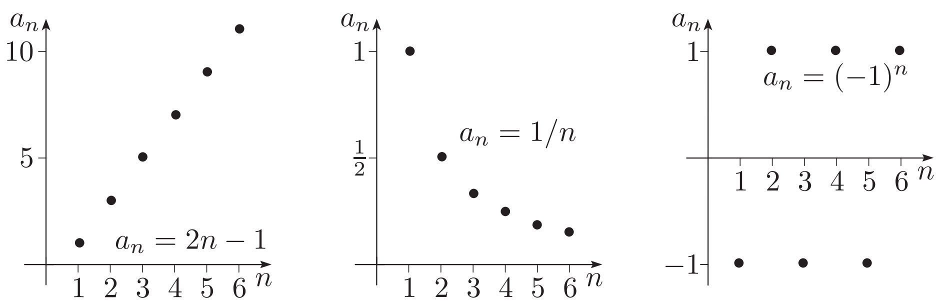


Figure 1 Three sequence diagrams

Exercise D23

Draw a sequence diagram, showing the first five points, for each of the following sequences (a_n).

(In part (c) you can use your solution to Exercise D22(e).)

(a) $a_n = n^2, \quad n = 1, 2, \ldots$

(b) $a_n = 3, \quad n = 1, 2, \ldots$

(c) $a_n = \left(1 + \dfrac{1}{n}\right)^n, \quad n = 1, 2, \ldots$

(d) $a_n = \dfrac{(-1)^n}{n}, \quad n = 1, 2, \ldots$

1.2 Monotonic sequences

Many sequences have the property that, as n increases, their terms are either *increasing* or *decreasing*. For example, the sequence $(2n-1)$ has terms $1, 3, 5, 7, \ldots$, which are increasing, whereas the sequence $(1/n)$ has terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, which are decreasing. The sequence $((-1)^n)$ is neither increasing nor decreasing. All this can be seen clearly on the sequence diagrams in Figure 1.

We now give precise meanings to these words *increasing* and *decreasing*, and introduce the word *monotonic*. These terms are illustrated in Figure 2.

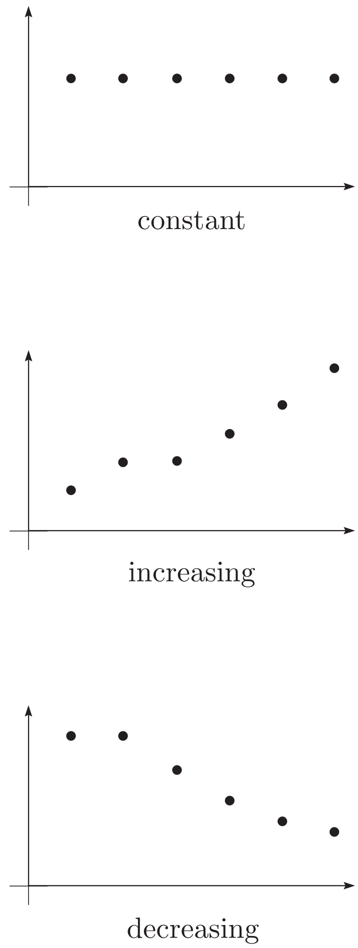


Figure 2 Monotonic sequences

Definitions

A sequence (a_n) is said to be

- **constant** if

$$a_{n+1} = a_n, \quad \text{for } n = 1, 2, \ldots;$$

- **increasing** if

$$a_{n+1} \geq a_n, \quad \text{for } n = 1, 2, \ldots,$$

  and **strictly increasing** if

$$a_{n+1} > a_n, \quad \text{for } n = 1, 2, \ldots;$$

- **decreasing** if

$$a_{n+1} \leq a_n, \quad \text{for } n = 1, 2, \ldots,$$

  and **strictly decreasing** if

$$a_{n+1} < a_n, \quad \text{for } n = 1, 2, \ldots;$$

- **monotonic** if (a_n) is either increasing or decreasing.

Notice that for a sequence (a_n) to be *increasing*, it is essential that $a_{n+1} \geq a_n$ for *all* $n \geq 1$. However, we do not require strict inequalities because we wish to describe a sequence such as

$$1, 1, 2, 2, 3, 3, 4, 4, \ldots$$

as increasing. If strict inequalities do hold, and we want to emphasise this, then we can use the term *strictly increasing* to describe the sequence, provided that $a_{n+1} > a_n$ for *all* $n \geq 1$. Thus every strictly increasing sequence is increasing, but the converse is not true. Similar comments apply to the terms *decreasing* and *strictly decreasing*. One slightly bizarre consequence of the definitions is that constant sequences are both increasing and decreasing (though not, of course, strictly increasing or strictly decreasing).

A sequence is *monotonic* if it is either increasing or decreasing (we do not require it to be strictly increasing or strictly decreasing, although it may be). To determine whether a given sequence is monotonic, it is not sufficient to draw a diagram: it is necessary to give a proof. There are various ways to do this. For example, $(1/n)$ is a strictly decreasing sequence and is therefore monotonic, because

$$\frac{1}{n+1} < \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots,$$

since $n + 1 > n > 0$, for $n = 1, 2, \ldots$. Here we have used Rule 4 for rearranging inequalities that you met in Unit D1 *Numbers*. We often use these rules when dealing with sequences, so for convenience they are restated in the box below.

Rules for rearranging inequalities

Let a, b, c and p be real numbers.

Rule 1 $a < b \iff b - a > 0$.

Rule 2 $a < b \iff a + c < b + c$.

Rule 3 If $c > 0$, then $a < b \iff ac < bc$;
if $c < 0$, then $a < b \iff ac > bc$.

Rule 4 If $a, b > 0$, then

$$a < b \iff \frac{1}{a} > \frac{1}{b}.$$

Rule 5 If $a, b \geq 0$ and $p > 0$, then

$$a < b \iff a^p < b^p.$$

Rule 6 $|a| < b \iff -b < a < b$.

All the above rules also hold if strict inequalities are replaced by weak inequalities.

The next worked exercise illustrates some of the other approaches that can be used to prove whether or not a sequence is monotonic.

Worked Exercise D19

Determine which of the following sequences (a_n) are monotonic.

(a) $a_n = \dfrac{1}{n}, \quad n = 1, 2, \ldots.$

(b) $a_n = (n-1)(n-2), \quad n = 1, 2, \ldots.$

(c) $a_n = (-1)^n, \quad n = 1, 2, \ldots.$

Solution

(a) You have just seen one proof that this sequence is monotonic. Here is another.

We have

$$a_n = 1/n \quad \text{and} \quad a_{n+1} = 1/(n+1),$$

so

$$\frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} < 1, \quad \text{for } n = 1, 2, \ldots.$$

It follows that

$$a_{n+1} < a_n, \quad \text{for } n = 1, 2, \ldots.$$

Thus (a_n) is strictly decreasing, so (a_n) is monotonic.

(b) For each $n \geq 1$, we have

$$a_n = (n-1)(n-2) = n^2 - 3n + 2$$

and

$$a_{n+1} = n(n-1) = n^2 - n,$$

so

$$a_{n+1} - a_n = 2n - 2 \geq 0, \quad \text{for } n = 1, 2, \ldots.$$

It follows that

$$a_{n+1} \geq a_n, \quad \text{for } n = 1, 2, \ldots.$$

Thus (a_n) is increasing, so (a_n) is monotonic.

Notice that this sequence is not strictly increasing, because $a_1 = a_2 = 0$.

(c) To prove that a sequence is not monotonic, use consecutive terms to show that the sequence is neither increasing nor decreasing.

Consider the first three terms of the sequence: $a_1 = -1$, $a_2 = 1$ and $a_3 = -1$.

We have $a_3 < a_2$, which means that (a_n) is not increasing. Also $a_2 > a_1$, which means that (a_n) is not decreasing.

Thus (a_n) is neither increasing nor decreasing and so is not monotonic.

Worked Exercise D19 illustrates the use of the following two strategies.

Strategy D3

To show that a given sequence (a_n) is monotonic, consider the difference $a_{n+1} - a_n$.

- If $a_{n+1} - a_n \geq 0$, for $n = 1, 2, \ldots$, then (a_n) is increasing.
- If $a_{n+1} - a_n \leq 0$, for $n = 1, 2, \ldots$, then (a_n) is decreasing.

If $a_n > 0$ for all n, then it is often more convenient to use the following strategy.

Strategy D4

To show that a given sequence (a_n) of *positive* terms is monotonic, consider the quotient $\dfrac{a_{n+1}}{a_n}$.

- If $\dfrac{a_{n+1}}{a_n} \geq 1$, for $n = 1, 2, \ldots$, then (a_n) is increasing.
- If $\dfrac{a_{n+1}}{a_n} \leq 1$, for $n = 1, 2, \ldots$, then (a_n) is decreasing.

For a positive sequence, you can use either strategy; which is best depends on whether you think it is easier to simplify the difference $a_{n+1} - a_n$ or the quotient a_{n+1}/a_n.

Exercise D24

Show that the following sequences (a_n) are monotonic.

(In part (a) remember that, by convention, $0! = 1$.)

(a) $a_n = (n-1)!, \quad n = 1, 2, \ldots$.

(b) $a_n = 2^{-n}, \quad n = 1, 2, \ldots$.

(c) $a_n = n + \dfrac{1}{n}, \quad n = 1, 2, \ldots$.

Often it is possible to *guess* whether or not a sequence defined by a formula is monotonic by calculating the first few terms. Consider, for example, the sequence (a_n) given by

$$a_n = \left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, \ldots .$$

In Exercise D22(e) you found that the first five terms of this sequence are approximately

$$2, 2.25, 2.37, 2.44, 2.49.$$

These terms suggest that the sequence (a_n) is increasing and, in fact, it is, as you will see in Section 5.

However, the first few terms of a sequence are not always a reliable guide to the sequence's behaviour. Consider, for example, the sequence

$$a_n = \frac{10^n}{n!}, \quad n = 1, 2, \ldots .$$

The first five terms of this sequence are approximately

$$10, 50, 167, 417, 833.$$

These terms suggest that (a_n) is increasing. However, calculation of more terms shows that this is not so, as you can see in Figure 3.

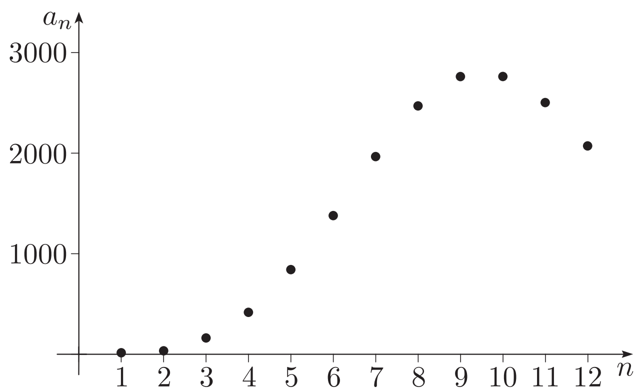


Figure 3 The sequence diagram for $a_n = 10^n/n!$

If we use Strategy D4, we find that

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10^{n+1}n!}{10^n(n+1)!} = \frac{10}{n+1}.$$

Now

$$\begin{aligned} \frac{10}{n+1} \leq 1 \quad &\Longleftrightarrow \quad n + 1 \geq 10 \\ &\Longleftrightarrow \quad n \geq 9. \end{aligned}$$

So

$$\frac{a_{n+1}}{a_n} \leq 1, \quad \text{for } n \geq 9.$$

Hence

$$a_{n+1} \leq a_n, \quad \text{for } n \geq 9,$$

so (a_n) is eventually decreasing (in fact, $a_9 = a_{10}$ and (a_n) is strictly decreasing for $n \geq 10$).

This type of situation arises quite often, so it is helpful to give a formal definition.

Definition

If a sequence (a_n) has a certain property provided we ignore a finite number of terms, we say that the sequence **eventually** has this property.

We have just seen that the sequence $(10^n/n!)$ is eventually decreasing. As another example of this usage, consider the sequence (a_n) defined by the formula

$$a_n = n^2, \quad n = 1, 2, \ldots .$$

Then we can say that the terms of this sequence are eventually greater than 100, because

$$n^2 > 100, \quad \text{for } n > 10.$$

Sometimes you may need to show that a sequence does *not* eventually have a certain property. To do this, you need to find infinitely many terms of the sequence which fail to have the property. For example, the terms of the sequence (a_n) defined by the formula

$$a_n = 3n, \quad n = 1, 2, \ldots .$$

are not eventually even, because $3n$ is an odd number whenever n is odd.

Exercise D25

Classify each of the following statements as true or false and justify your answers (if a statement is true, then prove it; if a statement is false, then explain why).

(a) The terms of the sequence (a_n) defined by $a_n = 2^n$ are eventually greater than 1000.

(b) The terms of the sequence (a_n) defined by $a_n = (-1)^n$ are eventually positive.

(c) The terms of the sequence (a_n) defined by $a_n = 1/n$ are eventually less than 0.025.

(d) The sequence (a_n) defined by $a_n = n^4/4^n$ is eventually decreasing.

2 Null sequences

In this section you will meet the definition of a *null sequence*, that is, a sequence which converges to 0. You will then explore the properties of null sequences and see how to prove these. You will also meet some basic null sequences and learn how to identify new null sequences.

2.1 What is a null sequence?

Let us begin by looking at the size of the terms of the sequence (a_n) defined by

$$a_n = \frac{1}{n}, \quad n = 1, 2, \ldots .$$

Worked Exercise D20

For each of the following statements about the terms of the above sequence (a_n), find an integer N that makes the statement true.

(a) $\dfrac{1}{n} < \dfrac{1}{100}$, for all $n > N$

(b) $\dfrac{1}{n} < \dfrac{3}{1000}$, for all $n > N$

Solution

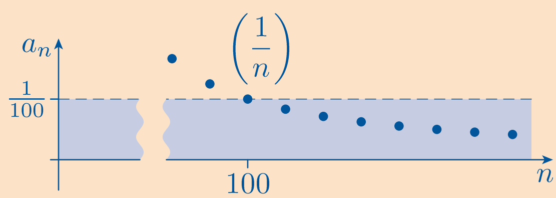
(a) Since the quantities involved are all positive, we can use Rule 4 to rearrange the inequality.

We have that

$$\frac{1}{n} < \frac{1}{100} \iff n > 100.$$

Hence we may take $N = 100$.

This is illustrated in the sequence diagram below.



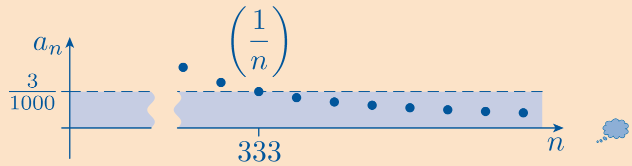
Of course, any integer greater than 100 is also a valid value for N, but any integer less than 100 is not.

(b) We have that

$$\frac{1}{n} < \frac{3}{1000} \iff n > 333.333\ldots .$$

Hence we may take $N = 333$.

The value $N = 333$ is valid since the smallest value of n that this allows is $n = 334$, and of course $334 > 333.333\ldots$. This is illustrated in the sequence diagram below.



The next exercise is similar to Worked Exercise D20, except that this time you are asked to look at a sequence with both positive and negative terms. The statements about the size of the terms therefore involve the *modulus* of the terms.

Exercise D26

For each of the following statements about the sequence (a_n) defined by the formula

$$a_n = \frac{(-1)^n}{n^2}, \quad n = 1, 2, \ldots,$$

find an integer N that makes the statement true.

(a) $\left|\frac{(-1)^n}{n^2}\right| < \frac{1}{100}$, for all $n > N$

(b) $\left|\frac{(-1)^n}{n^2}\right| < \frac{3}{1000}$, for all $n > N$

The solutions of Worked Exercise D20 and Exercise D26 suggest that, if (a_n) is a sequence defined by

$$a_n = \frac{1}{n}, \quad n = 1, 2, \ldots,$$

or by

$$a_n = \frac{(-1)^n}{n^2}, \quad n = 1, 2, \ldots,$$

then as n becomes larger and larger, the terms of the sequence get closer and closer to 0.

We need a formal way of describing precisely what we mean by this. In order to do this we introduce the Greek letter ε, pronounced 'epsilon', which we use to denote a positive number that may be as small as we please in any given particular instance.

The symbol ε was first used within proofs in analysis by Augustin-Louis Cauchy (1789–1857) in his *Cours d'Analyse* of 1821. Cauchy chose ε, which he also used in some of his work on probability, because it corresponds to the initial letter of erreur (error), a fact which seems rather amusing today given that ε is now the characteristic symbol of precision and rigour in analysis.

We see that, for each of the two sequences we are considering, the terms of the sequence eventually lie inside a horizontal strip in the sequence diagram from $-\varepsilon$ up to ε, and this is the case no matter how small ε is taken to be.

For the sequence defined by

$$a_n = \frac{1}{n}, \quad n = 1, 2, \ldots,$$

the sequence diagram is given again in Figure 4. In this case the terms of the sequence are all positive, so we need only look at a horizontal strip from 0 up to ε. In Worked Exercise D20 we considered the particular values $\varepsilon = 1/100$ and $\varepsilon = 3/1000$, but now we let ε represent *any* positive number, however small.

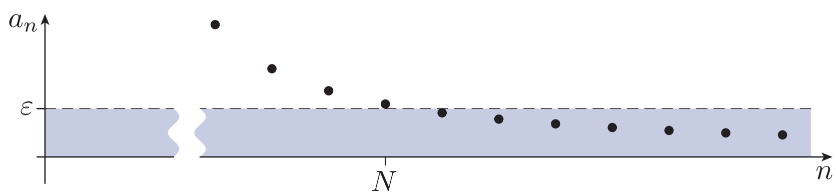


Figure 4 The sequence $a_n = 1/n$

This diagram suggests that, for each positive number ε, there is an integer N such that

$$|a_n| = a_n = \frac{1}{n} < \varepsilon, \qquad \text{for all } n > N.$$

This means that every term to the right of N in the diagram lies within the horizontal strip. In fact, this will be true if we take N to be any integer satisfying $N \geq \dfrac{1}{\varepsilon}$.

The sequence diagram in the case that

$$a_n = \frac{(-1)^n}{n^2}, \quad n = 1, 2, \ldots$$

is given in Figure 5.

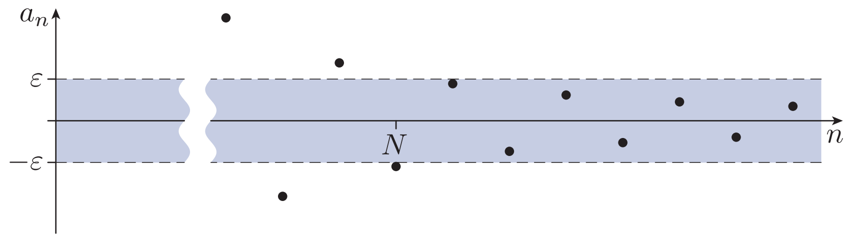


Figure 5 The sequence $a_n = \dfrac{(-1)^n}{n^2}$

This diagram suggests that, for each positive number ε, there is an integer N such that

$$|a_n| = \left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2} < \varepsilon, \quad \text{for all } n > N.$$

In fact, this will be true if we take N to be any integer satisfying $N \geq \sqrt{1/\varepsilon}$.

In both cases, the smaller we choose ε, the further to the right in the sequence diagram we have to go before we can be sure that all the terms of the sequence from that point onwards lie inside the strip. That is, the smaller we choose ε the larger we have to choose N if we wish to have

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

We now give a definition of a *null sequence* which formalises the notion of a sequence 'getting closer and closer to 0'. It follows from the discussion above that both of the sequences we have been considering are null sequences according to this definition. The concept of a null sequence is illustrated in Figure 6.

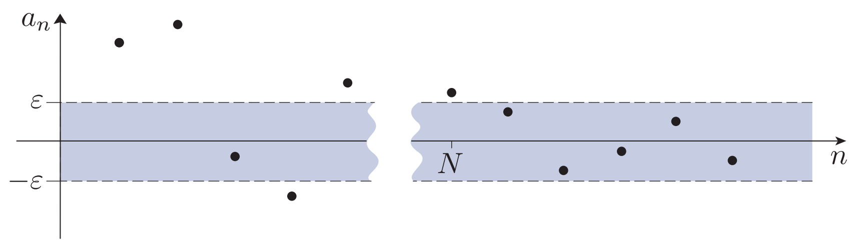


Figure 6 A null sequence

Definitions

The sequence (a_n) is **null** if

for each positive number ε, there is an integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

We also say that the sequence (a_n) is **convergent with limit 0**, or that (a_n) **converges** to 0.

Remarks

1. We write 'for all $n > N$' to emphasise that the inequality $|a_n| < \varepsilon$ holds for every integer $n > N$. Note that we can rewrite the last line of the definition as the implication

   $$\text{if } n > N, \text{ then } |a_n| < \varepsilon.$$

   We sometimes refer to this statement as the ε-N statement.
2. The sequence (a_n) is null if and only if the sequence $(|a_n|)$ is null. This is because the statement in the definition is identical for the sequences (a_n) and $(|a_n|)$.
3. The null sequence (a_n) remains null if we add, delete or alter a finite number of terms to produce a new sequence (b_n). Informally, we say that 'finitely many terms do not matter'.

   This is because the statement in the definition above and its corresponding version for (b_n) are identical, except that the values of N may differ by some integer.

We can interpret the task of finding a suitable integer N when using the definition as an 'ε-N game' in which player A chooses a positive number ε and challenges player B to find some integer N for which the statement in the definition is true. Thus, for example, it follows from Exercise D26 that, for the sequence (a_n) defined by

$$a_n = \frac{(-1)^n}{n^2}, \quad n = 1, 2, \ldots,$$

if player A chooses $\varepsilon = 1/100$ then player B can choose $N = 10$ (or any larger value). This is illustrated in Figure 7.

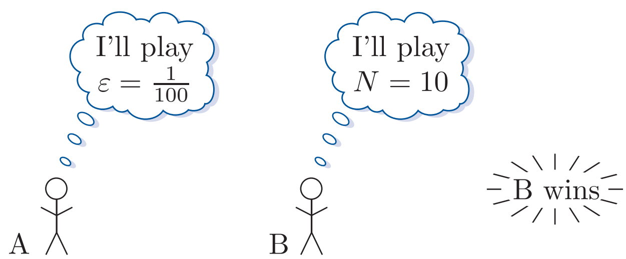


Figure 7 The ε-N game

Notice that in the 'ε-N game', if (a_n) is any null sequence and both players make their choices carefully, then player B will always win.

Worked Exercise D21

Prove that (a_n) is a null sequence if

$$a_n = \frac{1}{n^3}, \quad n = 1, 2, \ldots.$$

Solution

We have to prove that, for each $\varepsilon > 0$, there is an integer N such that

$$\left|\frac{1}{n^3}\right| < \varepsilon, \quad \text{for all } n > N. \qquad (*)$$

In order to find a suitable value of N, we rearrange the inequality in $(*)$ until we obtain an inequality with just n on one side. We use Rules 4 and 5 for rearranging inequalities.

We have that

$$\begin{aligned}\left|\frac{1}{n^3}\right| < \varepsilon &\iff \frac{1}{n^3} < \varepsilon \\ &\iff n^3 > \frac{1}{\varepsilon} \\ &\iff n > \frac{1}{\sqrt[3]{\varepsilon}}.\end{aligned}$$

So, statement $(*)$ holds if $N \geq \dfrac{1}{\sqrt[3]{\varepsilon}}$. Hence (a_n) is null.

Sometimes we might want to prove that a sequence (a_n) is *not* null. To do this, we have to show that (a_n) does not satisfy the definition of a null sequence. In other words, we must show that the following statement is true:

> there is some value of $\varepsilon > 0$ for which there is no integer N such that
>
> $$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

This is illustrated in the next worked exercise. Notice that in the 'ε-N game', if (a_n) is not a null sequence and both players make their choices carefully, then player A will always win.

Worked Exercise D22

Prove that the following sequence (a_n) is not null:

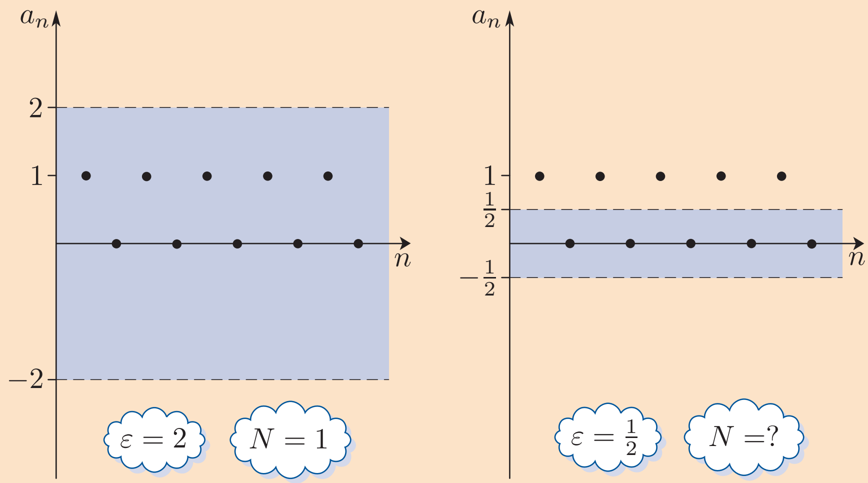
$$a_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Solution

We have to find a positive number ε for which there is no integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

In terms of the sequence diagram, this means that the sequence (a_n) does not eventually lie in the horizontal strip from $-\varepsilon$ to ε.



We can see that 2 would not be a suitable value of ε but $\frac{1}{2}$ would be a suitable value. In our 'ε-N game', if player A plays $\varepsilon = \frac{1}{2}$, then there is no integer N that player B can play to win.

For all odd values of n, we have $|a_n| = 1$ and so, if $\varepsilon = \frac{1}{2}$, there is no integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

Hence (a_n) is not a null sequence.

Notice that any positive value of ε less than 1 will serve our purpose here: there is nothing special about the number $\frac{1}{2}$.

Worked Exercises D21 and D22 illustrate the following strategy.

Strategy D5

- To show that the sequence (a_n) is null, rearrange the inequality $|a_n| < \varepsilon$ to find an integer N (generally depending on ε) such that $|a_n| < \varepsilon$, for all $n > N$.
- To show that the sequence (a_n) is not null, find *one* value of $\varepsilon > 0$ for which there is *no* integer N such that $|a_n| < \varepsilon$ for all $n > N$.

Exercise D27

Use Strategy D5 to determine which of the following sequences (a_n) are null.

(a) $a_n = \dfrac{1}{2n-1}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{(-1)^n}{10}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{(-1)^n}{n^4+1}, \quad n = 1, 2, \ldots$

Hint: You will need to consider the case where $\varepsilon \geq 1$ and the case where $0 < \varepsilon < 1$ separately.

2.2 Properties of null sequences

We now look at a number of properties of null sequences. These allow us to identify new null sequences without having to work with the definition.

This subsection contains several proofs; reading them should improve your understanding of the material. However, if you are short of time, you should skim through these proofs now and return to them when time permits.

Theorem D4 Power Rule for null sequences

If (a_n) is null, where $a_n \geq 0$, for $n = 1, 2, \ldots$, and p is a positive real number, then (a_n^p) is null.

Proof We want to prove that the sequence (a_n^p) is null; that is:

for each positive number ε, there is an integer N such that

$$a_n^p < \varepsilon, \quad \text{for all } n > N. \tag{1}$$

Here we use the fact that $|a_n^p| = a_n^p$, since $a_n \geq 0$.

We start by letting ε be a positive number. Since (a_n) is null and $\varepsilon^{1/p}$ is positive, there is an integer N such that

$$a_n < \varepsilon^{1/p}, \quad \text{for all } n > N. \tag{2}$$

Taking the pth power of both sides of the inequality in statement (2), we see that statement (1) holds with the same value of N. ■

Note how we used $\varepsilon^{1/p}$ in statement (2) in order to obtain ε in statement (1). We often prove the ε-N statement for some new null sequence by applying the definition to a known null sequence (or sequences), using a positive number related in a suitable way to ε.

Earlier we saw that the sequence (a_n) defined by

$$a_n = \frac{1}{n}, \quad n = 1, 2, \ldots$$

is null. By applying the Power Rule with $p = 3$, we can now deduce that the sequence (b_n) defined by

$$b_n = \frac{1}{n^3}, \quad n = 1, 2, \ldots$$

is also null.

We will use the next set of rules a great deal.

Theorem D5 Combination Rules for null sequences

If (a_n) and (b_n) are null, then:

Sum Rule $(a_n + b_n)$ is null

Multiple Rule (λa_n) is null, for any real number λ

Product Rule $(a_n b_n)$ is null.

Proof We first prove the Sum Rule. We want to prove that the sequence $(a_n + b_n)$ is null; that is:

for each positive number ϵ, there is an integer N such that

$$|a_n + b_n| < \varepsilon, \quad \text{for all } n > N. \tag{3}$$

Let ε be a positive number. Since (a_n) and (b_n) are null, there are integers N_1 and N_2 such that

$$|a_n| < \tfrac{1}{2}\varepsilon, \quad \text{for all } n > N_1, \quad \text{and} \quad |b_n| < \tfrac{1}{2}\varepsilon, \quad \text{for all } n > N_2.$$

We use $\frac{1}{2}\varepsilon$ here in order to obtain ε in statement (3).

If $N = \max\{N_1, N_2\}$, then both the above inequalities hold for all $n > N$. Therefore, by the Triangle Inequality (which you met in Unit D1),

$$|a_n + b_n| \le |a_n| + |b_n| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \quad \text{for all } n > N.$$

Thus statement (3) holds with this value of N.

Next, we prove the Multiple Rule. We want to prove that the sequence (λa_n) is null; that is:

for each positive number ε, there is an integer N such that

$$|\lambda a_n| < \varepsilon, \quad \text{for all } n > N. \tag{4}$$

If $\lambda = 0$, this statement is obvious, so we can assume that $\lambda \neq 0$.

Let ε be a positive number. Since (a_n) is null, there is an integer N such that

$$|a_n| < \varepsilon/|\lambda|, \quad \text{for all } n > N.$$

We use $\varepsilon/|\lambda|$ here in order to obtain ε in statement (4).

Multiplying both sides of this inequality by the positive number $|\lambda|$ gives

$$|\lambda a_n| < \varepsilon, \quad \text{for all } n > N.$$

Thus statement (4) holds with this value of N.

Finally, we prove the Product Rule. We want to prove that the sequence (a_nb_n) is null; that is:

for each positive number ε, there is an integer N such that

$$|a_nb_n| < \varepsilon, \quad \text{for all } n > N. \tag{5}$$

Let ε be a positive number. Since (a_n) and (b_n) are null, there are integers N_1 and N_2 such that

$$|a_n| < \sqrt{\varepsilon}, \quad \text{for all } n > N_1, \quad \text{and} \quad |b_n| < \sqrt{\varepsilon}, \quad \text{for all } n > N_2.$$

We use $\sqrt{\varepsilon}$ here in order to obtain ε in statement (5).

If $N = \max\{N_1, N_2\}$, then both the above inequalities hold for all $n > N$, so if we multiply them we obtain

$$|a_nb_n| = |a_n||b_n| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon, \quad \text{for all } n > N.$$

Thus statement (5) holds with this value of N. ■

Worked Exercise D23

Use the Power and Combination Rules, and any sequences that you have already shown to be null, to show that the following sequences (a_n) are null.

(a) $a_n = \dfrac{1}{n} + \dfrac{1}{n^3}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{6}{\sqrt[5]{n}} + \dfrac{5}{(2n-1)^7}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{1}{n(2n-1)}, \quad n = 1, 2, \ldots$

Solution

(a) We know that the sequences $\left(\dfrac{1}{n}\right)$ and $\left(\dfrac{1}{n^3}\right)$ are null, so (a_n) is null, by the Sum Rule.

(b) The sequences $\left(\dfrac{1}{n}\right)$ and $\left(\dfrac{1}{2n-1}\right)$ are null, so the sequences $\left(\dfrac{6}{\sqrt[5]{n}}\right)$ and $\left(\dfrac{5}{(2n-1)^7}\right)$ are null, by the Power Rule and the Multiple Rule.

Hence (a_n) is null, by the Sum Rule.

(c) We know that $\left(\dfrac{1}{n}\right)$ and $\left(\dfrac{1}{2n-1}\right)$ are null, so (a_n) is null, by the Product Rule.

Exercise D28

Use the Power and Combination Rules, and any sequences that you have already shown to be null, to show that the following sequences (a_n) are null.

(a) $a_n = \dfrac{1}{(2n-1)^3}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{7\pi}{n^3}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{1}{3n^4(2n-1)^{1/3}}, \quad n = 1, 2, \ldots$

Our next rule, the Squeeze Rule, also enables us to 'get new null sequences from old' – but in a slightly different way. To illustrate this rule, we look first at the sequence diagrams of two sequences (a_n) and (b_n) defined by

$$a_n = \frac{1}{1+\sqrt{n}}, \quad n = 1, 2, \ldots$$

and

$$b_n = \frac{1}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$

We know that (b_n) is a null sequence by the Power Rule, but what about (a_n)?

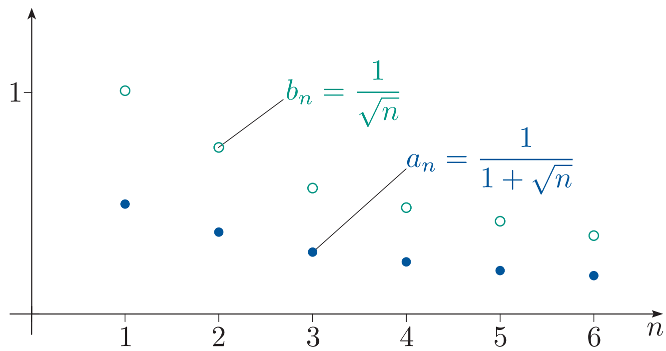


Figure 8 The sequence diagrams for $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$

Figure 8 shows that the points corresponding to the sequence (a_n) are 'squeezed' in between the horizontal axis and the points corresponding to the null sequence (b_n), since

$$0 < \frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

We express this in words by saying that the sequence (a_n) is **dominated** by the sequence (b_n). Hence, if from some point onwards all the points corresponding to (b_n) lie in a narrow horizontal strip in the sequence diagram from $-\varepsilon$ up to ε, then (from the same point onwards) all the points corresponding to (a_n) will also lie in the same strip. So, since ε may be any positive number, it certainly looks from this sequence diagram argument that (a_n) must be a null sequence too.

Theorem D6 Squeeze Rule for null sequences

If (b_n) is a null sequence of non-negative terms, and

$$|a_n| \leq b_n, \quad \text{for } n = 1, 2, \ldots,$$

then (a_n) is null.

Proof We want to prove that (a_n) is null; that is:

for each positive number ε, there is an integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N. \qquad (6)$$

Let ε be a positive number. Then since (b_n) is null and its terms are non-negative, there is some integer N such that

$$|b_n| = b_n < \varepsilon, \quad \text{for all } n > N.$$

We also know that $|a_n| \leq b_n$, for $n = 1, 2, \ldots$, so it follows that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

Thus statement (6) holds with this value of N. ■

To show that a sequence is null using the Squeeze Rule, we use the following strategy.

Strategy D6

To use the Squeeze Rule to show that a sequence (a_n) is null, do the following.

1. Guess a dominating null sequence (b_n) with non-negative terms.
2. Check that $|a_n| \leq b_n$, for $n = 1, 2, \ldots$.

The following worked exercise illustrates the use of the strategy.

Worked Exercise D24

Use the Squeeze Rule to show that the following sequences (a_n) are null.

(a) $a_n = \dfrac{(-1)^n}{n^3 + 1}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{2\cos(2n)}{n^2}, \quad n = 1, 2, \ldots$

Solution

(a) To guess a suitable dominating sequence, we look at the formula for a_n and try to spot a related null sequence whose terms have larger magnitude and are non-negative. Here, $(1/n^3)$ seems to be a likely candidate.

We guess that (a_n) is dominated by (b_n), where

$$b_n = \frac{1}{n^3}, \quad n = 1, 2, \ldots.$$

To check this, we have to show that

$$\left| \frac{(-1)^n}{n^3+1} \right| \leq \frac{1}{n^3}, \quad \text{for } n = 1, 2, \ldots.$$

This holds because

$$n^3 + 1 \geq n^3, \quad \text{for } n = 1, 2, \ldots.$$

We showed earlier that (b_n) is null, so we can deduce that (a_n) is null, by the Squeeze Rule.

(b) In this case, we know that $-1 \leq \cos(2n) \leq 1$ by the properties of the cosine function. This suggests that $(2/n^2)$ might be a suitable dominating sequence.

We guess that (a_n) is dominated by (b_n), where

$$b_n = \frac{2}{n^2}, \quad n = 1, 2, \ldots.$$

To check this, we have to show that

$$\left| \frac{2\cos(2n)}{n^2} \right| \leq \frac{2}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

This holds because

$$|\cos(2n)| \leq 1, \quad \text{for } n = 1, 2, \ldots.$$

We showed earlier that the sequence $(1/n)$ is null, so it follows from the Power Rule and the Multiple Rule that (b_n) is null. We deduce that (a_n) is null, by the Squeeze Rule.

Exercise D29

Use the Squeeze Rule to show that the following sequences (a_n) are null.

(a) $a_n = \dfrac{1}{n^2+n}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{(-1)^n}{n!}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{\sin(n^2)}{n^2+2^n}, \quad n = 1, 2, \ldots$

2.3 Basic null sequences

We now show that there are various basic types of sequences that are null. By applying the rules from the previous subsection to these 'basic null sequences', we can deduce the existence of many different null sequences without having to use the definition.

It is important that you are familiar with these types of basic null sequences and are able to use them. Reading the proof that they are null may help you with this, but skim read it if you are short of time and return to it when time permits.

Theorem D7 Basic null sequences

The following sequences are null.

(a) $(1/n^p)$, for $p > 0$.

(b) (c^n), for $|c| < 1$.

(c) $(n^p c^n)$, for $p > 0, |c| < 1$.

(d) $(c^n/n!)$, for $c \in \mathbb{R}$.

(e) $(n^p/n!)$, for $p > 0$.

Proof (a) To prove that $(1/n^p)$ is null for $p > 0$, we apply the Power Rule to the sequence $(1/n)$, which we know is null.

(b) To prove that (c^n) is null for $|c| < 1$, first note that it is sufficient to consider only the case $0 \leq c < 1$, because any sequence (a_n) is null if and only if the sequence $(|a_n|)$ is null.

If $c = 0$, then the sequence is obviously null. Thus we can assume that $0 < c < 1$, so we can write

$$c = \frac{1}{1+a}, \quad \text{where } a > 0.$$

Expressing c in this way enables us to use the Binomial Theorem from Unit D1, which says that, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(1+x)^n = 1 + nx + \tfrac{1}{2}n(n-1)x^2 + \cdots + x^n.$$

Here we put $x = a$, and since a is positive, so is every term on the right-hand side.

By the Binomial Theorem

$$(1+a)^n \geq 1 + na \geq na, \quad \text{for } n = 1, 2, \ldots,$$

so

$$c^n = \frac{1}{(1+a)^n} \leq \frac{1}{na}, \quad \text{for } n = 1, 2, \ldots.$$

Since $(1/n)$ is null, we deduce that $(1/(na))$ is null, by the Multiple Rule. Hence (c^n) is null, by the Squeeze Rule, as required.

(c) To prove that $(n^p c^n)$ is null, for $p > 0$ and $|c| < 1$, we can again assume that $0 < c < 1$, so

$$c = \frac{1}{1+a}, \quad \text{where } a > 0.$$

First we deal with the case $p = 1$; that is, we consider the sequence (nc^n).

We use the Binomial Theorem again, but this time include a further term in the expansion.

By the Binomial Theorem,

$$(1+a)^n \geq 1 + na + \tfrac{1}{2}n(n-1)a^2 \geq \tfrac{1}{2}n(n-1)a^2, \quad \text{for } n = 2, 3, \ldots,$$

so

$$nc^n = \frac{n}{(1+a)^n} \leq \frac{n}{\frac{1}{2}n(n-1)a^2} = \frac{(2/a^2)}{n-1}, \quad \text{for } n = 2, 3, \ldots.$$

Now the sequence (a_n) defined by

$$a_n = \frac{(2/a^2)}{n-1}, \quad n = 2, 3, \ldots$$

is the same as the sequence defined by

$$a_n = \frac{(2/a^2)}{n}, \quad n = 1, 2, \ldots,$$

so (a_n) is null by the Multiple Rule. Hence (nc^n) is null, by the Squeeze Rule. This proves part (c) in the case $p = 1$.

To deduce that $(n^p c^n)$ is null for any $p > 0$ and $0 < c < 1$, we note that

$$n^p c^n = (nd^n)^p, \quad \text{for } n = 1, 2, \ldots,$$

where $d = c^{1/p}$.

Notice that the sequence (nd^n) is in a form that enables us to apply part (c) in the case $p = 1$, which we have just proved.

Since $0 < d < 1$, we know that (nd^n) is null, so $(n^p c^n)$ is null for any $p > 0$, by the Power Rule.

(d) To prove that $(c^n/n!)$ is null, we can again assume that $c > 0$. We first choose an integer m such that $m + 1 > c$. Then, for $n > m + 1$,

$$\begin{aligned}\frac{c^n}{n!} &= \left(\frac{c}{1}\right)\left(\frac{c}{2}\right)\cdots\left(\frac{c}{m}\right)\left(\frac{c}{m+1}\right)\cdots\left(\frac{c}{n-1}\right)\left(\frac{c}{n}\right)\\ &\leq \left(\frac{c}{1}\right)\left(\frac{c}{2}\right)\cdots\left(\frac{c}{m}\right) \times \frac{c}{n}\\ &= K \times \frac{c}{n},\end{aligned}$$

where $K = c^m/m!$ is a constant.

Here we have used that fact that, since $c < m + 1$, it follows that $c/d < 1$ for any $d \geq m + 1$.

Since $(1/n)$ is null, we deduce that (Kc/n) is null, by the Multiple Rule. Hence $(c^n/n!)$ is null, by the Squeeze Rule.

(e) To prove that $(n^p/n!)$ is null for $p > 0$, we write

$$\frac{n^p}{n!} = \left(\frac{n^p}{2^n}\right)\left(\frac{2^n}{n!}\right), \quad \text{for } n = 1, 2, \ldots.$$

Since $(n^p/2^n)$ is a null sequence, by part (c) with $c = 1/2$, and $(2^n/n!)$ is also null, by part (d) with $c = 2$, we deduce that $(n^p/n!)$ is null, by the Product Rule. ■

Exercise D30

Verify that each of the following sequences (a_n) is a basic null sequence by identifying its type from those listed in Theorem D7, giving the values of c and/or p in each case.

(a) $a_n = 0.9^n, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{27^n}{n!}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{1}{\sqrt{n}}, \quad n = 1, 2, \ldots$

(d) $a_n = \dfrac{n^{27}}{n!}, \quad n = 1, 2, \ldots$

(e) $a_n = n\left(\dfrac{1}{2}\right)^n, \quad n = 1, 2, \ldots$

3 Convergent sequences

In the previous section we looked at null sequences, that is, sequences which converge to 0. We now turn our attention to sequences which converge to limits other than 0.

3.1 What is a convergent sequence?

The following exercise should help to give you some understanding of the behaviour of a sequence which has a limit that is not 0.

Exercise D31

Consider the sequence

$$a_n = \frac{n+1}{n}, \quad n = 1, 2, \ldots .$$

(a) Draw the sequence diagram of (a_n) and describe (informally) how this sequence behaves.

(b) What can you say (formally) about the behaviour of the sequence

$$b_n = a_n - 1, \quad n = 1, 2, \ldots ?$$

The terms of the sequence (a_n) in Exercise D31 appear to approach arbitrarily close to 1; that is, the sequence (a_n) appears to converge to 1. If we subtract 1 from each term a_n to form the sequence (b_n), then we obtain a null sequence. This example suggests the following definition of a *convergent sequence*.

Definitions

The sequence (a_n) is **convergent** with **limit $\boldsymbol{l}$** if $(a_n - l)$ is a null sequence. We say that (a_n) **converges to $\boldsymbol{l}$** and we write

either $\lim_{n\to\infty} a_n = l$

or $a_n \to l$ as $n \to \infty$.

Remarks

1. The statements in the definition are read as:
   'the limit of a_n, as n tends to infinity, is l';
   'a_n tends to l, as n tends to infinity'.
2. Often we omit 'as $n \to \infty$'.
3. Do not let this use of the *symbol* ∞ tempt you to think that ∞ is a real number. Instead, you should remember that the phrase 'a_n tends to l, as n tends to infinity' means that 'as n gets larger and larger, a_n gets closer and closer to l'.

The following are examples of convergent sequences:

- every null sequence converges to 0
- every constant sequence (c) converges to c
- as you saw in Exercise D31, the sequence (a_n) defined by

  $$a_n = \frac{n+1}{n}, \quad n = 1, 2, \ldots$$

  is convergent and $\lim_{n\to\infty} a_n = 1$.

Exercise D32

Show that the sequence

$$a_n = \frac{n^3+1}{2n^3}, \quad n = 1, 2, \ldots$$

converges to $\frac{1}{2}$, by considering $a_n - \frac{1}{2}$.

The definition of convergence of a sequence is often given in the following equivalent (alternative) form, mirroring the definition of a null sequence given in the previous section.

Definition (alternative)

The sequence (a_n) **converges to l** if

for each positive number ε, there is an integer N such that

$$|a_n - l| < \varepsilon, \quad \text{for all } n > N.$$

Remarks

1. In terms of the sequence diagram for (a_n), this definition states that:

   for each positive number ε, the terms a_n eventually lie inside the horizontal strip from $l - \varepsilon$ to $l + \varepsilon$.

   This is illustrated in Figure 9.

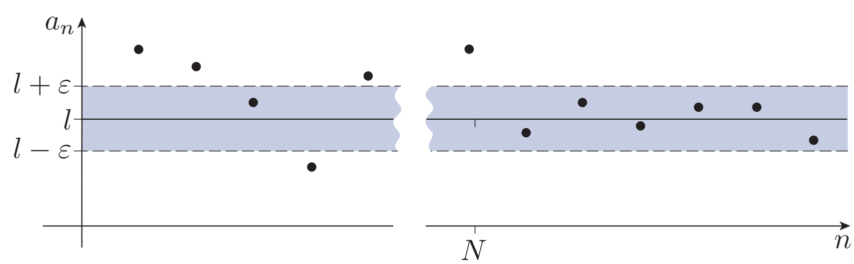


Figure 9 A sequence which converges to l

2. If a sequence is convergent, then it has a unique limit. A proof of this seemingly obvious fact is given later in this section.
3. If a given sequence converges to l, then this remains true if we add, delete or alter a finite number of terms. This follows from the corresponding result for null sequences.
4. Not all sequences are convergent, as you will see in Section 4. For example, the sequence $((-1)^n)$ is not convergent.

3.2 Combination Rules for convergent sequences

So far you have tested the convergence of a given sequence (a_n) by calculating $a_n - l$ and showing that $(a_n - l)$ is null. This presupposes that you know in advance the value of l. Usually, however, you are given a sequence (a_n) and asked to decide whether or not it converges and, if it does, to *find* its limit. Fortunately, the convergence of many sequences can be proved by using the following Combination Rules, which extend the Combination Rules for null sequences.

Theorem D8 Combination Rules for convergent sequences

If $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$, then:

Sum Rule $\lim_{n\to\infty}(a_n + b_n) = l + m$

Multiple Rule $\lim_{n\to\infty}(\lambda a_n) = \lambda l$, for $\lambda \in \mathbb{R}$

Product Rule $\lim_{n\to\infty}(a_n b_n) = lm$

Quotient Rule $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{l}{m}$, provided that $m \neq 0$.

In applications of the Quotient Rule, some terms b_n can take the value 0, in which case a_n/b_n is not defined. However, we shall see (in Lemma D9) that because $m \neq 0$ this occurs for only *finitely many* b_n, so (b_n) is eventually non-zero. Thus the statement of the Quotient Rule does make sense.

We prove the Combination Rules at the end of this subsection, but first we illustrate how to apply them. When using these rules, there is no need for you to identify which particular rule you are using: you can refer simply to the Combination Rules.

Applying the Combination Rules

It is straightforward to apply the Combination Rules to simple sums, multiples, products or quotients of sequences that you already know are convergent. For example, since you know that $(1/n^2)$ is a basic null sequence, and that the constant sequence (c) has limit c, you can deduce from the Sum Rule that

$$\lim_{n\to\infty}\left(c + \frac{1}{n^2}\right) = c + 0 = c,$$

and from the Quotient Rule that

$$\lim_{n\to\infty}\left(\frac{1}{cn^2}\right) = \frac{0}{c} = 0.$$

However, the real power of the Combination Rules is that they enable us to determine the limits of some more complicated sequences, whose nth term is given by a quotient that at first sight may not seem to involve convergent sequences. This is illustrated in the next worked exercise.

Worked Exercise D25

Show that each of the following sequences (a_n) is convergent and find its limit.

(a) $a_n = \dfrac{(2n+1)(n+2)}{3n^2+3n}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{2n^2+10^n}{n!+3n^3}, \quad n = 1, 2, \ldots$

Solution

Although the expressions for a_n are quotients, we cannot apply the Quotient Rule immediately because the sequences defined by the numerators and the denominators are not convergent. In each case, however, we can rearrange the expressions for a_n and then apply the Combination Rules.

(a) Dividing both the numerator and the denominator by n^2 gives

$$a_n = \frac{(2n+1)(n+2)}{3n^2+3n} = \frac{(2+1/n)(1+2/n)}{3+3/n}.$$

Since $(1/n)$ is a basic null sequence, we find, by the Combination Rules, that

$$\lim_{n\to\infty} a_n = \frac{(2+0)(1+0)}{3+0} = \frac{2}{3}.$$

(b) Dividing both the numerator and the denominator by $n!$ gives

$$a_n = \frac{2n^2+10^n}{n!+3n^3} = \frac{2n^2/n!+10^n/n!}{1+3n^3/n!}.$$

Since $(n^2/n!)$, $(10^n/n!)$ and $(n^3/n!)$ are all basic null sequences, we find, by the Combination Rules, that (a_n) is convergent and

$$\lim_{n\to\infty} a_n = \frac{0+0}{1+0} = 0.$$

In Worked Exercise D25 the key step in determining the limit of a given sequence was to rearrange the expression for a_n in a way that enabled us to apply the Combination Rules. To explain how this is done, we need the following definition.

Definition

The **dominant term** of a quotient involving the variable n, where $n = 1, 2, \ldots$, is the term in n (without its coefficient) which eventually has the largest absolute value.

As a simple example, consider the quotient

$$\frac{n^3 + 1}{2n^3}, \quad n = 1, 2, \ldots,$$

which is the formula for the nth term of the sequence you met in Exercise D32. In this quotient there are two terms in n, namely n^3 and $2n^3$. To find the dominant term we exclude the coefficients, so both terms reduce to n^3; hence, n^3 is the dominant term in this quotient.

The method used to rearrange the expressions for a_n in Worked Exercise D25 was to divide both the numerator and the denominator by the dominant term of the quotient. Doing this converts the quotient into a form where the Combination Rules can be applied.

- In part (a) the dominant term is n^2, which is the highest power of n in the quotient. We then used the fact that $(1/n)$ is a null sequence to find the limit of (a_n) using the Combination Rules.
- In part (b) the dominant term is $n!$, because $n!$ eventually becomes larger than n^2, n^3 and 10^n as n increases. We then used the fact that $(n^2/n!)$, $(10^n/n!)$ and $(n^3/n!)$ are all basic null sequences to find the limit of (a_n) using the Combination Rules.

These examples illustrate the following general strategy, where the list of dominant terms follows from the list of basic null sequences.

Strategy D7

To evaluate the limit of a sequence whose nth term is a complicated quotient, do the following.

1. Identify the dominant term, noting that

   $$n! \text{ dominates } c^n,$$

   and, for $|c| > 1$ and $p > 0$,

   $$c^n \text{ dominates } n^p.$$

2. Divide both numerator and denominator by the dominant term.
3. Apply the Combination Rules.

Exercise D33

Show that each of the following sequences (a_n) is convergent and find its limit.

(a) $a_n = \dfrac{n^3 + 2n^2 + 3}{2n^3 + 1}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{n^2 + 2^n}{3^n + n^3}, \quad n = 1, 2, \ldots$

Hint: You can use the fact that $(2^n/3^n)$ is a basic null sequence, because $2^n/3^n = (2/3)^n$, for $n = 1, 2, \ldots$.

(c) $a_n = \dfrac{n! + (-1)^n}{2^n + 3n!}, \quad n = 1, 2, \ldots$

Proofs of the Combination Rules

We now prove the Sum Rule, the Multiple Rule and the Product Rule by using the corresponding Combination Rules for null sequences.

Remember that $\lim_{n\to\infty} a_n = l$ means that $(a_n - l)$ is a null sequence.

Sum Rule for convergent sequences

If $\lim\limits_{n\to\infty} a_n = l$ and $\lim\limits_{n\to\infty} b_n = m$, then

$$\lim_{n\to\infty} (a_n + b_n) = l + m.$$

Proof We know that $(a_n - l)$ and $(b_n - m)$ are null sequences. Since

$$(a_n + b_n) - (l + m) = (a_n - l) + (b_n - m),$$

we deduce that $((a_n + b_n) - (l + m))$ is null, by the Sum Rule for null sequences. ■

We now prove the Product Rule. The Multiple Rule is a special case of the Product Rule in which the sequence (b_n) is a constant sequence.

Product Rule for convergent sequences

If $\lim\limits_{n\to\infty} a_n = l$ and $\lim\limits_{n\to\infty} b_n = m$, then

$$\lim_{n\to\infty} (a_n b_n) = lm.$$

Proof Here we express $a_n b_n - lm$ in terms of $a_n - l$ and $b_n - m$:

$$a_n b_n - lm = (a_n - l)(b_n - m) + m(a_n - l) + l(b_n - m).$$

Since $(a_n - l)$ and $(b_n - m)$ are null, we deduce that $(a_n b_n - lm)$ is null, by the Combination Rules for null sequences. ■

To prove the Quotient Rule we use the following lemma, which shows that if the limit of a sequence is positive, then the terms of the sequence must eventually be positive.

Lemma D9

If $\lim_{n\to\infty} a_n = l$ and $l > 0$, then there is an integer N such that

$$a_n > \tfrac{1}{2}l, \quad \text{for all } n > N.$$

Proof By taking $\varepsilon = \frac{1}{2}l$ in the alternative definition of convergence from Subsection 3.1, we see that there is an integer N such that

$$|a_n - l| < \tfrac{1}{2}l, \quad \text{for all } n > N.$$

This is illustrated in Figure 10.

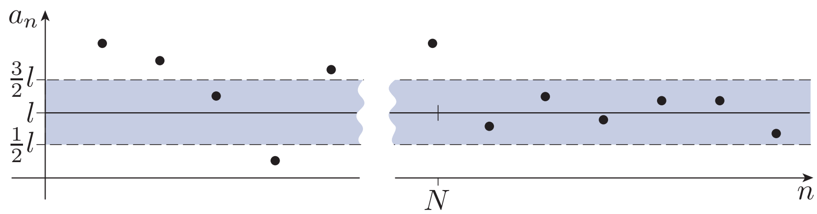


Figure 10 The sequence diagram for (a_n)

Hence

$$-\tfrac{1}{2}l < a_n - l < \tfrac{1}{2}l, \quad \text{for all } n > N,$$

and the left-hand inequality gives

$$\tfrac{1}{2}l < a_n, \quad \text{for all } n > N,$$

as required. ■

We now prove the Quotient Rule, which completes the proof of Theorem D8.

Quotient Rule for convergent sequences

If $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{l}{m}, \quad \text{provided that } m \neq 0.$$

Proof We give the proof for $m > 0$; the proof for the case $m < 0$ is similar. Once again the idea is to write the required expression in terms of $a_n - l$ and $b_n - m$:

$$\frac{a_n}{b_n} - \frac{l}{m} = \frac{m(a_n - l) - l(b_n - m)}{b_n m}.$$

Now, however, there is a slight problem: $(m(a_n - l) - l(b_n - m))$ is certainly a null sequence, but the denominator $b_n m$ is rather awkward. Some of the terms b_n can take the value 0, in which case the expression is undefined.

However, by Lemma D9, we know that for some integer N we have

$$b_n > \tfrac{1}{2}m, \quad \text{for all } n > N,$$

so the terms of (b_n) are eventually positive. Thus, for all $n > N$,

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{l}{m}\right| &= \frac{|m(a_n - l) - l(b_n - m)|}{b_n m}\\ &\leq \frac{|m(a_n - l) - l(b_n - m)|}{\frac{1}{2}m^2}.\end{aligned}$$

Since the right-hand side defines a null sequence, it follows, by the Squeeze Rule for null sequences, that $\left(\dfrac{a_n}{b_n} - \dfrac{l}{m}\right)$ is null, as required. ■

3.3 Further rules for convergent sequences

There are several other theorems about convergent sequences, which are needed in later units. The first is a general version of the Squeeze Rule, illustrated in Figure 11.

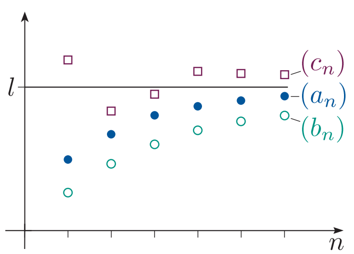


Figure 11 The Squeeze Rule

Theorem D10 Squeeze Rule for convergent sequences

If (a_n), (b_n) and (c_n) are sequences such that

1. $b_n \leq a_n \leq c_n$, for $n = 1, 2, \ldots,$
2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = l$,

then $\lim\limits_{n\to\infty} a_n = l$.

Proof By the Combination Rules,

$$\lim_{n\to\infty} (c_n - b_n) = l - l = 0,$$

so $(c_n - b_n)$ is a null sequence. Also, by condition 1 in the statement of the theorem,

$$0 \leq a_n - b_n \leq c_n - b_n, \quad \text{for } n = 1, 2, \ldots,$$

so $(a_n - b_n)$ is null, by the Squeeze Rule for null sequences.

Now we write a_n in the form

$$a_n = (a_n - b_n) + b_n.$$

Then, by the Combination Rules,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_n - b_n) + \lim_{n\to\infty} b_n = 0 + l = l.$$

■

Note that in applications of the Squeeze Rule, it is sufficient to check that condition 1 *eventually* holds. This is because the values of a *finite* number of terms do not affect convergence.

The following worked exercise and exercise illustrate the use of the Squeeze Rule and the Binomial Theorem in the derivation of two important limits.

Worked Exercise D26

(a) Prove that, if $c > 0$, then

$$(1+c)^{1/n} \le 1 + \frac{c}{n}, \quad \text{for } n = 1, 2, \ldots.$$

(b) Use the Squeeze Rule to deduce that if $a > 0$, then

$$\lim_{n\to\infty} a^{1/n} = 1.$$

Solution

(a) We proved this inequality for the case $c = 1$ in Worked Exercise D13 of Unit D1.

Using Rule 5 for rearranging inequalities with $p = n$, we obtain

$$(1+c)^{1/n} \le 1 + \frac{c}{n} \iff 1 + c \le \left(1 + \frac{c}{n}\right)^n.$$

The right-hand inequality holds because

$$\left(1 + \frac{c}{n}\right)^n \ge 1 + n\left(\frac{c}{n}\right) = 1 + c,$$

by the Binomial Theorem, so the left-hand inequality also holds.

(b) We consider the cases $a > 1$, $a = 1$ and $0 < a < 1$ separately.

If $a > 1$, then we can write $a = 1 + c$, where $c > 0$.

In this application of the Squeeze Rule we take the 'lower' sequence to be the constant sequence whose terms are all equal to 1.

By part (a),

$$1 < a^{1/n} = (1+c)^{1/n} \le 1 + \frac{c}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Since $(1/n)$ is a basic null sequence, it follows from the Combination Rules that $\lim_{n\to\infty}\left(1 + \frac{c}{n}\right) = 1$. We deduce, by the Squeeze Rule, that

$$\lim_{n\to\infty} a^{1/n} = 1.$$

If $a = 1$, then $a^{1/n} = 1$, for $n = 1, 2, \ldots$, so

$$\lim_{n\to\infty} a^{1/n} = 1.$$

Finally, if $0 < a < 1$, then $1/a > 1$, so $\lim_{n\to\infty} (1/a)^{1/n} = 1$, by the first case. Hence, by the Quotient Rule,

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \frac{1}{(1/a)^{1/n}} = \frac{1}{1} = 1.$$

Exercise D34

(a) Prove that

$$n^{1/n} \leq 1 + \sqrt{\frac{2}{n-1}}, \quad \text{for } n \geq 2.$$

Hint: By the Binomial Theorem, we have

$$(1+x)^n \geq \frac{n(n-1)}{2!}x^2, \quad \text{for } n \geq 2,\ x \geq 0.$$

(b) Use the Squeeze Rule to deduce from part (a) that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Next we show that taking limits preserves weak inequalities.

Theorem D11 Limit Inequality Rule

If $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$, and also

$$a_n \leq b_n, \quad \text{for } n = 1, 2, \ldots,$$

then $l \leq m$.

Proof We use proof by contradiction. Suppose that $a_n \to l$, $b_n \to m$ and $a_n \leq b_n$, for $n = 1, 2, \ldots$. If $l > m$, then, by the Combination Rules,

$$\lim_{n\to\infty} (a_n - b_n) = l - m > 0.$$

Hence, by Lemma D9, there is an integer N such that

$$a_n - b_n > \tfrac{1}{2}(l-m), \quad \text{for all } n > N. \tag{7}$$

Since $a_n - b_n \leq 0$, for $n = 1, 2, \ldots$, statement (7) gives a contradiction.

Hence the inequality $l \leq m$ is true. ■

We have the following corollary, promised earlier, that a convergent sequence has a unique limit.

Corollary D12

If $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = m$, then $l = m$.

Proof Applying the Limit Inequality Rule with $b_n = a_n$, we deduce that $l \leq m$ and also that $m \leq l$. Hence $l = m$. ■

Note that taking limits does not preserve *strict* inequalities. For example, if $a_n = 1/n$, $n = 1, 2, \ldots$, and $b_n = 2/n$, $n = 1, 2, \ldots$, then

$$a_n < b_n, \quad \text{for } n = 1, 2, \ldots.$$

But it is not true that $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$, since both limits are 0; this is illustrated in Figure 12.

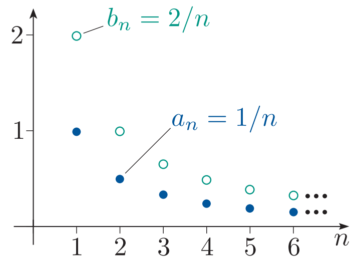


Figure 12 Two sequences with the same limit

In Subsection 2.1, we pointed out that a sequence (a_n) is null if and only if the sequence $(|a_n|)$ is null. The final theorem in this section is a partial generalisation of this result.

Theorem D13

If $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} |a_n| = |l|$.

Proof We want to show that $(|a_n| - |l|)$ is null. Using the backwards form of the Triangle Inequality, which you met in Unit D1, we obtain

$$\big||a_n| - |l|\big| \leq |a_n - l|, \quad \text{for } n = 1, 2, \ldots.$$

Since $(a_n - l)$ is null, so is $(|a_n - l|)$, and we deduce from the Squeeze Rule for null sequences that $(|a_n| - |l|)$ is null, as required. ■

Note that Theorem D13 is only a partial generalisation of the earlier result about null sequences because its converse is false: if $|a_n| \to |l|$, then it does *not* follow that $a_n \to l$. For example, consider the sequence $a_n = (-1)^n$, $n = 1, 2, \ldots$; in this case,

$$|a_n| \to 1 \text{ as } n \to \infty,$$

but (a_n) does not converge.

4 Divergent sequences

In previous sections you have seen many examples of sequences that are convergent. We now investigate the behaviour of sequences which do not converge.

4.1 What is a divergent sequence?

Any sequence that does not converge is said to be *divergent*.

Definition

A sequence is **divergent** if it is not convergent.

Figure 13 gives the sequence diagrams for three different sequences (a_n). Each of these sequences is divergent but, as you can see, they behave differently.

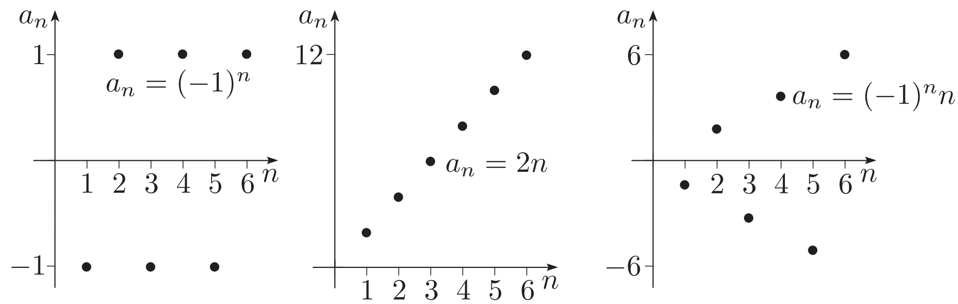


Figure 13 Three divergent sequences

It is not easy to prove from the definition that these sequences are divergent. To show that a sequence (a_n) is divergent, we would have to show that (a_n) is not convergent; that is, for *every* real number l, the sequence $(a_n - l)$ is not null.

In this section we obtain criteria for divergence which avoid us having to argue directly from the definition. At the end of the section we give a strategy for proving divergence using two criteria, which together cover all cases. We obtain these criteria by establishing certain properties which are necessarily possessed by a convergent sequence; if a sequence does *not* have one of these properties, then it must be divergent.

4.2 Bounded and unbounded sequences

One property possessed by a convergent sequence is that it must be *bounded*, as we will show.

Definitions

A sequence (a_n) is **bounded** if there is a number M such that

$$|a_n| \leq M, \quad \text{for } n = 1, 2, \ldots.$$

A sequence is **unbounded** if it is not bounded.

Thus a sequence (a_n) is bounded if *all* the terms a_n lie on the sequence diagram in the horizontal strip from $-M$ to M, for some positive number M; see Figure 14.

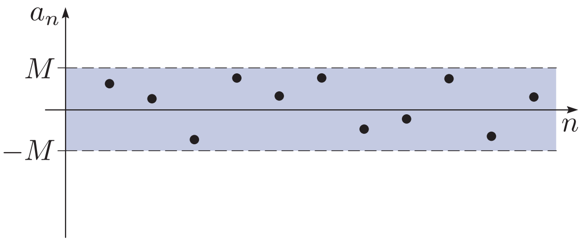


Figure 14 A bounded sequence

For example, the sequence $((-1)^n)$ is bounded because

$$|(-1)^n| \leq 1, \quad \text{for } n = 1, 2, \ldots.$$

However, the sequences $(2n)$ and (n^2) are unbounded since, for each positive number M, we can find terms of these sequences whose absolute values are greater than M.

Exercise D35

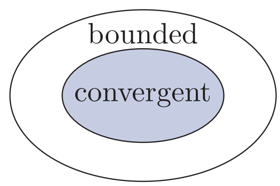
Classify the following sequences (a_n) as bounded or unbounded.

(a) $a_n = 1 + (-1)^n, \quad n = 1, 2, \ldots$

(b) $a_n = (-1)^n n, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{2n+1}{n}, \quad n = 1, 2, \ldots$

The sequence $((-1)^n)$ shows that a bounded sequence is not necessarily convergent. However, we can prove that a convergent sequence is necessarily bounded. This is illustrated in Figure 15.



convergent $\Longrightarrow$ bounded

Figure 15 Convergent and bounded sequences

Theorem D14

If (a_n) is convergent, then (a_n) is bounded.

Proof We know that $a_n \to l$, for some real number l, so $(a_n - l)$ is a null sequence. Taking $\varepsilon = 1$ in the definition of a null sequence, we see that there is an integer N such that

$$|a_n - l| < 1, \quad \text{for all } n > N.$$

Now

$$\begin{aligned} |a_n| &= |(a_n - l) + l| \\ &\leq |a_n - l| + |l|, \quad \text{by the Triangle Inequality.} \end{aligned}$$

It follows that

$$|a_n| < 1 + |l|, \quad \text{for all } n > N.$$

This is the type of inequality needed to prove that (a_n) is bounded, but it does not include the terms $a_1, a_2, \ldots, a_N$. To complete the proof, we put

$$M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |l|\}.$$

It then follows that

$$|a_n| \leq M, \quad \text{for } n = 1, 2, \ldots,$$

as required. ■

From Theorem D14 we obtain the following test for the *divergence* of a sequence, which is illustrated in Figure 16.

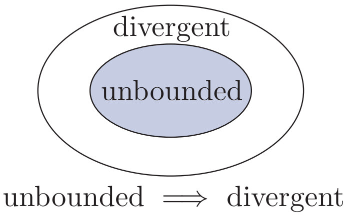


Figure 16 Unbounded and divergent sequences

Corollary D15

If (a_n) is unbounded, then (a_n) is divergent.

For example, the sequences $(2n)$ and $((-1)^n n)$ are both unbounded, so they are both divergent, by Corollary D15.

Exercise D36

Classify the following sequences (a_n) as bounded or unbounded and as convergent or divergent.

(a) $a_n = \sqrt{n}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{n^2 + n}{n^2 + 1}, \quad n = 1, 2, \ldots$

(c) $a_n = (-1)^n n^2, \quad n = 1, 2, \ldots$

(d) $a_n = n^{(-1)^n}, \quad n = 1, 2, \ldots$

4.3 Sequences tending to infinity

Although the sequences $(2n)$ and $((-1)^n n)$ are both unbounded and hence divergent, there is a marked difference in their behaviour. Informally, the terms of both sequences become arbitrarily large, but those of the sequence $(2n)$ become arbitrarily large and positive. The following definition makes this informal idea precise.

Definition

The sequence (a_n) **tends to infinity** if

for each positive number M, there is an integer N such that

$$a_n > M, \quad \text{for all } n > N.$$

In this case, we write

$$a_n \to \infty \text{ as } n \to \infty.$$

Remarks

1. Often we omit 'as $n \to \infty$' and simply write $a_n \to \infty$.
2. In terms of the sequence diagram for (a_n), this definition states that, for each positive number M, the terms a_n eventually lie above the horizontal line at height M; see Figure 17.

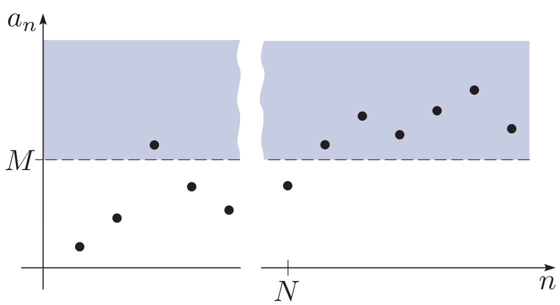


Figure 17 A sequence tending to infinity

3. If a sequence tends to infinity, then it is unbounded and hence divergent, by Corollary D15.
4. If a given sequence tends to infinity, then this remains true if we add, delete or alter a finite number of terms.

The next rule enables us to use our knowledge of null sequences to identify sequences which tend to infinity.

Theorem D16 Reciprocal Rule for sequences

If the sequence (a_n) satisfies the conditions

1. (a_n) is eventually positive
2. $(1/a_n)$ is a null sequence

then $a_n \to \infty$.

Proof To prove that $a_n \to \infty$, we have to show that:

for each positive number M, there is an integer N such that

$$a_n > M, \quad \text{for all } n > N. \tag{8}$$

Let M be a positive number. Since (a_n) is eventually positive, we can choose an integer N_1 such that

$$a_n > 0, \quad \text{for all } n > N_1.$$

Since $(1/a_n)$ is null, we can take $\varepsilon = 1/M$ in the definition of a null sequence and choose an integer N_2 such that

$$\left|\frac{1}{a_n}\right| < \frac{1}{M}, \quad \text{for all } n > N_2.$$

Now let $N = \max\{N_1, N_2\}$; then

$$0 < \frac{1}{a_n} < \frac{1}{M}, \quad \text{for all } n > N.$$

This statement is equivalent to statement (8), so $a_n \to \infty$. ■

The next worked exercise illustrates the use of the Reciprocal Rule.

Worked Exercise D27

Use the Reciprocal Rule to prove that the following sequences (a_n) tend to infinity.

(a) $a_n = n^3/2, \quad n = 1, 2, \ldots$

(b) $a_n = n! + 10^n, \quad n = 1, 2, \ldots$

(c) $a_n = n! - 10^n, \quad n = 1, 2, \ldots$

Solution

(a) Each term of the sequence (a_n) is positive and

$$\frac{1}{a_n} = \frac{2}{n^3}.$$

Now $(1/n^3)$ is a basic null sequence, so $(2/n^3)$ is null, by the Multiple Rule.

Hence $a_n \to \infty$, by the Reciprocal Rule.

(b) Each term of the sequence $(n! + 10^n)$ is positive.

The dominant term is $n!$, so we write

$$\frac{1}{a_n} = \frac{1}{n! + 10^n} = \frac{1/n!}{1 + 10^n/n!}.$$

Now, $(1/n!)$ and $(10^n/n!)$ are basic null sequences. Thus, by the Combination Rules,

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0}{1+0} = 0.$$

Alternatively, you could argue that, since $\dfrac{1}{n! + 10^n} \leq \dfrac{1}{n!}$, the sequence $\left(\dfrac{1}{n! + 10^n}\right)$ is null by the Squeeze Rule for null sequences.

It follows that $a_n \to \infty$, by the Reciprocal Rule.

(c) The first few terms of this sequence are *not* positive but we can show that (a_n) is *eventually* positive.

The dominant term is $n!$, so we first write

$$n! - 10^n = n!(1 - 10^n/n!), \quad n = 1, 2, \ldots.$$

Since $(10^n/n!)$ is a basic null sequence, we know that $10^n/n!$ is eventually less than 1, so $(n! - 10^n)$ is eventually positive.

Next we write

$$\frac{1}{a_n} = \frac{1}{n! - 10^n} = \frac{1/n!}{1 - 10^n/n!}.$$

Now, $(1/n!)$ and $(10^n/n!)$ are basic null sequences. Thus, by the Combination Rules,

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0}{1-0} = 0.$$

Hence $a_n \to \infty$, by the Reciprocal Rule.

There are also versions of the Combination Rules and Squeeze Rule for sequences which tend to infinity. We state these without proof. Recall that $\mathbb{R}^+$ is the set of positive real numbers; that is, $\mathbb{R}^+ = \{x : x > 0\}$.

Theorem D17 Combination Rules for sequences which tend to infinity

If (a_n) tends to infinity and (b_n) tends to infinity, then

Sum Rule $(a_n + b_n)$ tends to infinity

Multiple Rule (λa_n) tends to infinity, for $\lambda \in \mathbb{R}^+$

Product Rule $(a_n b_n)$ tends to infinity.

Theorem D18 Squeeze Rule for sequences which tend to infinity

If (b_n) tends to infinity and

$$a_n \geq b_n, \quad \text{for } n = 1, 2, \ldots,$$

then (a_n) tends to infinity.

Exercise D37

For each of the following sequences (a_n), prove that $a_n \to \infty$.

(a) $a_n = 2^n/n, \quad n = 1, 2, \ldots$

(b) $a_n = 2^n - n^9, \quad n = 1, 2, \ldots$

(c) $a_n = 2^n/n + 5n^9, \quad n = 1, 2, \ldots$

(d) $a_n = \dfrac{2^n + n^2}{n^9 + n}, \quad n = 1, 2, \ldots$

We can also define what it means for a sequence (a_n) to *tend to minus infinity*. (Note that, in some texts, the symbol $+\infty$ is used for sequences that tend to ∞, in order to have symmetry with the symbol $-\infty$.)

Definition

The sequence (a_n) **tends to minus infinity** if

$$-a_n \to \infty \quad \text{as } n \to \infty.$$

We write

$$a_n \to -\infty \quad \text{as } n \to \infty.$$

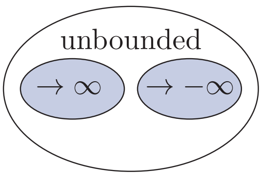


Figure 18 Unbounded sequences

For example, the sequence $(-n^2)$ tends to minus infinity because (n^2) tends to infinity. Sequences which tend to minus infinity are unbounded and hence divergent. However, the sequence $((-1)^n n)$ shows that an unbounded sequence need not tend to either infinity or to minus infinity; see Figure 18.

4.4 Subsequences

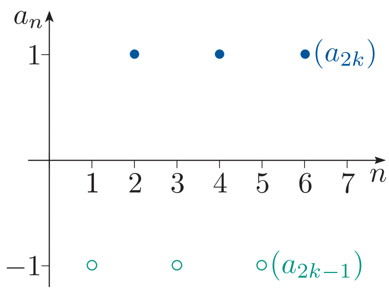


Figure 19 The sequence $a_n = ((-1)^n)$

In this subsection we give two more criteria for establishing that a sequence diverges; both involve the idea of a *subsequence*. For example, consider the bounded divergent sequence $((-1)^n)$. This sequence splits naturally into two:

the *even* terms $a_2, a_4, \ldots, a_{2k}, \ldots,$ each of which equals 1

the *odd* terms $a_1, a_3, \ldots, a_{2k-1}, \ldots,$ each of which equals -1.

Both of these are sequences in their own right, and we call them the **even subsequence** (a_{2k}) and the **odd subsequence** (a_{2k-1}). This is illustrated in Figure 19.

In general, given a sequence (a_n), we can consider many different subsequences, such as:

(a_{3k}), comprising the terms $a_3, a_6, a_9, \ldots$

(a_{4k+1}), comprising the terms $a_5, a_9, a_{13}, \ldots$

$(a_{k!})$, comprising the terms $a_1, a_2, a_6, \ldots$.

We assume that k takes the values from 1 to ∞ unless specified otherwise.

Definition

The sequence (a_{n_k}) is a **subsequence** of the sequence (a_n) if (n_k) is a strictly increasing sequence of positive integers; that is,

$$n_1 < n_2 < n_3 < \cdots .$$

The subsequence (a_{n_k}) of the sequence (a_n) has terms

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$$

and is often specified by a formula giving n_k in terms of k. For example, the subsequence $(a_{n_k}) = (a_{5k+2})$ corresponds to those terms from (a_n) whose subscripts are given by positive integers

$$n_k = 5k + 2, \quad k = 1, 2, \ldots .$$

Thus the first term of (a_{5k+2}) is a_7, the second is a_{12}, and so on.

Note that if (n_k) is any strictly increasing sequence of positive integers, then $n_k \geq k$, for $k = 1, 2, \ldots$, so $n_k \to \infty$ as $k \to \infty$.

Exercise D38

(a) Let $a_n = n^2$, $n = 1, 2, \ldots$. Write down the first five terms of each of the subsequences (a_{n_k}), where:

(i) $n_k = 2k$ (ii) $n_k = 4k - 1$ (iii) $n_k = k^2$.

(b) Write down the first three terms of the odd and even subsequences of the sequence (a_n) defined by

$$a_n = n^{(-1)^n}, \quad n = 1, 2, \ldots .$$

Next we show that certain properties of sequences are inherited by their subsequences.

Theorem D19

For any subsequence (a_{n_k}) of a sequence (a_n):

(a) if $a_n \to l$ as $n \to \infty$, then $a_{n_k} \to l$ as $k \to \infty$

(b) if $a_n \to \infty$ as $n \to \infty$, then $a_{n_k} \to \infty$ as $k \to \infty$.

Proof We prove only part (a); the proof of part (b) is similar. We want to show that

for each positive number ε, there is a positive integer K such that
$$|a_{n_k} - l| < \varepsilon, \quad \text{for all } k > K. \tag{9}$$

Let ε be a positive number. Since $(a_n - l)$ is null, we know that there is a positive integer N such that

$$|a_n - l| < \varepsilon, \quad \text{for all } n > N.$$

If we take K so large that $n_K \geq N$, then

$$n_k > n_K \geq N, \quad \text{for all } k > K.$$

Hence statement (9) holds for this value of K, as required. ■

The following criteria for establishing that a sequence is *divergent* are immediate consequences of Theorem D19(a).

Corollary D20 Subsequence Rules

First Subsequence Rule The sequence (a_n) is divergent if (a_n) has two convergent subsequences with different limits.

Second Subsequence Rule The sequence (a_n) is divergent if (a_n) has a subsequence which tends to infinity or a subsequence which tends to minus infinity.

We can now formulate a general strategy for showing that a sequence is divergent, as promised at the beginning of this section.

Strategy D8

To prove that the sequence (a_n) is divergent, either:

- show that (a_n) has two convergent subsequences with different limits, or
- show that (a_n) has a subsequence which tends to infinity or a subsequence which tends to minus infinity.

For example, the sequence $((-1)^n)$ has two convergent subsequences which have different limits, namely, the even subsequence with limit 1 and the odd subsequence with limit -1. So the sequence $((-1)^n)$ is divergent, by the First Subsequence Rule.

On the other hand, the sequence $\left(n^{(-1)^n}\right)$ has a subsequence (the even subsequence) which tends to infinity since, if $n = 2k$, then $n^{(-1)^n} = 2k$. So $\left(n^{(-1)^n}\right)$ is divergent, by the Second Subsequence Rule.

To apply Strategy D8 successfully, you need to be able to recognise convergent subsequences with different limits, or a subsequence (which may be the whole sequence) which tends to infinity or to minus infinity. It is not always easy to do this, and some experimentation may be required. If the formula for a_n involves the expression $(-1)^n$, it is a good idea to consider the odd and even subsequences, although this may not always work. It may be helpful to calculate the values of the first few terms in order to try to identify suitable subsequences.

Exercise D39

Use Strategy D8 to prove that each of the following sequences (a_n) is divergent. Remember that $\lfloor x \rfloor$ denotes the integer part of x.

(a) $a_n = (-1)^n + \dfrac{1}{n}, \quad n = 1, 2, \ldots$

(b) $a_n = \frac{1}{3}n - \lfloor \frac{1}{3}n \rfloor, \quad n = 1, 2, \ldots$

(c) $a_n = n \sin\left(\frac{1}{2}n\pi\right), \quad n = 1, 2, \ldots$

We end this section by giving a result about subsequences which will be needed in later analysis units.

We say that a sequence (a_n) *consists* of two subsequences (a_{m_k}) and (a_{n_k}) when every term of the sequence appears in one or other of the subsequences. For example, every sequence (a_n) consists of its even subsequence (a_{2k}) and its odd subsequence (a_{2k-1}). The next theorem tells us that, in these circumstances, if the two subsequences tend to the same limit, then so does the whole sequence.

Theorem D21

Let (a_n) consist of two subsequences (a_{m_k}) and (a_{n_k}), which both tend to the *same* limit l. Then

$$\lim_{n\to\infty} a_n = l.$$

Proof We want to show that

for each $\varepsilon > 0$, there is an integer N such that

$$|a_n - l| < \varepsilon, \quad \text{for all } n > N. \tag{10}$$

Let ε be a positive number. We know that there are integers K_1 and K_2 such that

$$|a_{m_k} - l| < \varepsilon, \quad \text{for all } k > K_1,$$

and

$$|a_{n_k} - l| < \varepsilon, \quad \text{for all } k > K_2.$$

Now let

$$N = \max\{m_{K_1}, n_{K_2}\}.$$

Since each $n > N$ is either of the form m_k, with $k > K_1$, or of the form n_k, with $k > K_2$, we deduce that statement (10) holds with this value of N. ■

5 Monotone Convergence Theorem

In this section you will see a proof of the Monotone Convergence Theorem, which states that any increasing sequence which is bounded above must be convergent.

We illustrate the theorem with particular sequences that converge to π and to e. These applications of the Monotone Convergence Theorem are not assessed, but you should make sure that you have a good understanding of the statement of the theorem as we will use it in later analysis units.

5.1 Convergence of monotonic sequences

In Section 3 you met various techniques for finding the limit of a convergent sequence. As a result, you may be under the impression that, if we know that a sequence converges, then we can always find its limit. However, it is sometimes possible to prove that a sequence is convergent without being able to find its limit.

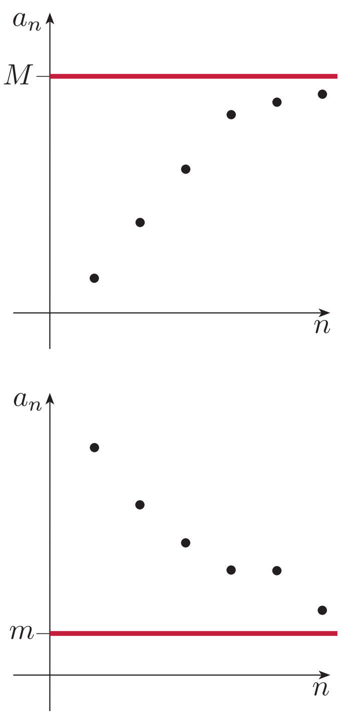


Figure 20 An increasing sequence that is bounded above and a decreasing sequence that is bounded below

For example, this situation can occur with a given sequence (a_n) with the following two properties:

1. (a_n) is an *increasing* sequence
2. (a_n) is *bounded above*; that is, there is a real number M such that

$$a_n \leq M, \quad \text{for } n = 1, 2, \ldots .$$

We will prove that such a sequence must be convergent. Likewise, if (a_n) is a sequence which is *decreasing* and *bounded below*, then (a_n) must be convergent. These ideas are illustrated in Figure 20.

The next theorem combines these results.

Theorem D22 Monotone Convergence Theorem

If the sequence (a_n) is either

- increasing and bounded above, or
- decreasing and bounded below,

then (a_n) is convergent.

Proof We prove only that (a_n) is convergent if it is increasing and bounded above; the proof where (a_n) is decreasing and bounded below is similar.

Since (a_n) is bounded above, the set $\{a_n : n = 1, 2, \ldots\}$ has a least upper bound, l say.

This follows from the Least Upper Bound Property of $\mathbb{R}$, which you met in Subsection 4.3 of Unit D1. The sequence (a_n) is illustrated in Figure 21.

We now prove that

$$\lim_{n\to\infty} a_n = l.$$

We want to show that

for each $\varepsilon > 0$, there is an integer N such that
$$|a_n - l| < \varepsilon, \quad \text{for all } n > N. \tag{11}$$

Let ε be a positive number. Since l is the least upper bound of the set $\{a_n : n = 1, 2, \ldots\}$, there is an integer N such that

$$a_N > l - \varepsilon.$$

If this were not true, then $l - \varepsilon$ would be an upper bound of the set $\{a_n : n = 1, 2, \ldots\}$, contradicting the fact that l is the *least* upper bound.

Because (a_n) is increasing, we have $a_n \geq a_N$ for $n > N$, so

$$a_n > l - \varepsilon, \quad \text{for all } n > N.$$

But then, since $a_n \leq l$ for all n, it follows that

$$|a_n - l| = l - a_n < \varepsilon, \quad \text{for all } n > N,$$

which proves statement (11). Hence (a_n) converges to l. ■

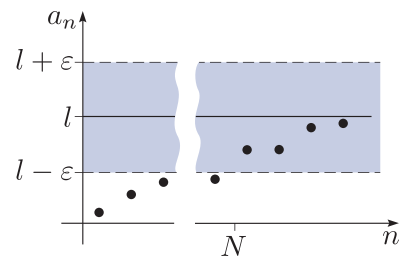


Figure 21 An increasing sequence with least upper bound l

The Monotone Convergence Theorem tells us that a sequence such as $(1 - 1/n)$, which is increasing and bounded above (by 1, for example), must be convergent. In this case, of course, we already know that $(1 - 1/n)$ is convergent with limit 1, without using the Monotone Convergence Theorem.

The Monotone Convergence Theorem is most useful when we suspect that a sequence is convergent, but we cannot find the limit directly. It can also be used to give precise definitions of numbers, such as π, about which we have only an informal idea, as you will see in the next subsection.

For completeness, we point out that if (a_n) is increasing but is *not* bounded above, then $a_n \to \infty$. For if (a_n) is not bounded above then, for any real number M, we can find an integer N such that $a_N > M$. Since (a_n) is increasing, we have $a_n \geq a_N$ for $n > N$, so

$$a_n > M, \quad \text{for all } n > N.$$

Hence $a_n \to \infty$ as $n \to \infty$.

Similarly, if (a_n) is decreasing but is not bounded below, then $a_n \to -\infty$.

We now summarise all these results about monotonic sequences.

Theorem D23 Monotonic Sequence Theorem

If the sequence (a_n) is monotonic, then either (a_n) is convergent or $a_n \to \pm\infty$.

Exercise D40

Prove that if the sequence (a_n) is increasing and has a subsequence (a_{n_k}) which is convergent, then (a_n) is convergent.

5.2 The number π

The rest of this section is not assessed but is included for your interest.

One of the oldest mathematical problems is to determine the area and the length of the perimeter of a disc of radius r. It is well known that these quantities are given by the formulas πr^2 and $2\pi r$, respectively. But what exactly is π?

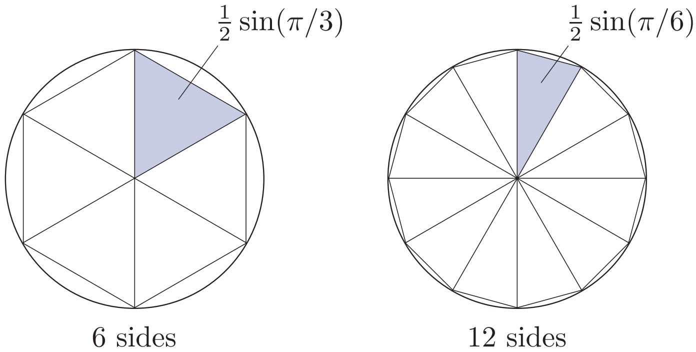
We define π by giving a precise definition of the area of a disc of radius 1. Our definition is based on a method used by Archimedes to approximate a circle of radius 1 by regular polygons circumscribed and inscribed in the circle.

Archimedes (c.287–c.212 BCE) of Syracuse was one of the greatest scientists of classical antiquity. Although many details of his life have survived, they are largely anecdotal and should be treated with caution. He had an interest in many areas of mathematics: geometry, arithmetic, astronomy, mechanics, statics (levers and centres of gravity), and hydrostatics (bodies floating in water). In particular, he anticipated modern calculus and analysis by applying the Eudoxean method of exhaustion to derive and rigorously prove a range of geometrical results including the area of a circle, the surface area and volume of a sphere and the area under a parabola. He is also credited with the design of many mechanical inventions including several war machines for use against the Roman armies laying siege to Syracuse.

What is especially noticeable about Archimedes, by comparison with many earlier mathematicians, is the way he combined pure geometrical analysis, and the mechanical or practical: his work on the lever is purely geometrical, whereas his astronomical book *On Sphere-making* (now lost) was about constructing a planetarium that modelled the motions of heavenly bodies. Indeed, his reputation in the centuries after his death was more as a maker of mechanical marvels than as a geometer.

Archimedes established bounds for the value of π by taking a circle of radius 1 and considering the perimeters of circumscribed and inscribed polygons, starting with a regular hexagon (6 sides) and progressively doubling the number of sides so as to get regular polygons of 12, 24, 48 and 96 sides, becoming ever closer to the circle. Using this method he calculated the value of π to lie between $3\frac{10}{71}$ and $3\frac{1}{7}$.

Here we will use the areas of the polygons used by Archimedes instead of their perimeters, but the details are similar. The areas of the inscribed polygons give a lower estimate for the area of the disc and hence for π. By doubling the number of sides of the polygon, we improve the estimate. The areas of the polygons can be calculated quite simply, as illustrated in Figure 22.



$$\begin{aligned}\text{area} &= 6 \times \tfrac{1}{2}\sin(\pi/3)\\ &= 6 \times \tfrac{1}{4}\sqrt{3}\\ &\approx 2.598\end{aligned} \qquad\qquad \begin{aligned}\text{area} &= 12 \times \tfrac{1}{2}\sin(\pi/6)\\ &= 12 \times \tfrac{1}{4}\\ &= 3\end{aligned}$$

Figure 22 The area of regular inner polygons

Let s_n denote the number of sides of the nth such inner polygon, so $s_1 = 6$, $s_2 = 12$ and, in general, $s_n = 3 \times 2^n$. The nth inner polygon consists of s_n isosceles triangles, each with two equal sides of length 1 that meet at an angle $2\pi/s_n$. Thus the total area a_n of the polygon is given by

$$a_n = \tfrac{1}{2}s_n \sin(2\pi/s_n), \quad \text{for } n = 1, 2, \ldots. \tag{12}$$

For example (to 3 d.p.) we have

$$a_1 = 2.598, \quad a_2 = 3, \quad \ldots, \quad a_6 = 3.141.$$

Geometrically, it is clear that each time we double the number of sides of the inner polygon, the area increases, so

$$a_1 < a_2 < a_3 < \cdots < a_n < a_{n+1} < \cdots .$$

Hence the sequence (a_n) is (strictly) increasing.

Note that each of the polygons lies inside a square of side 2, which has area 4; see Figure 23. This implies that

$$a_n \leq 4, \quad \text{for } n = 1, 2, \ldots.$$

Thus the sequence (a_n) is bounded above by 4.

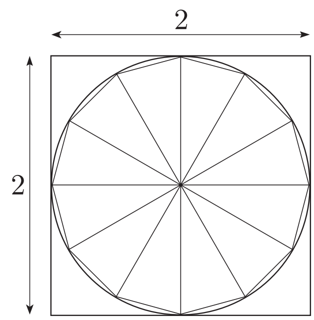


Figure 23 A circle inscribed in a square

Hence, by the Monotone Convergence Theorem, the sequence (a_n) is convergent, with limit at most 4. Our intuitive idea of the area of the disc suggests that it is greater than each of the areas a_n, but 'only just'. We know that the area of the circle in which the polygons are inscribed is equal to π, since it has radius 1. This leads us to make the following definition.

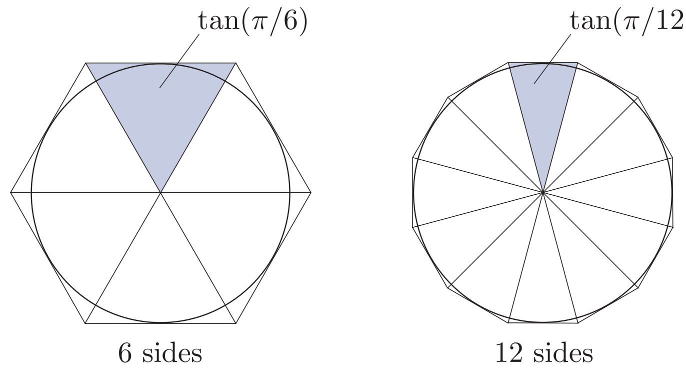
Definition

$$\pi = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \tfrac{1}{2} s_n \sin(2\pi/s_n),$$

where $s_n = 3 \times 2^n$.

We will explain in a moment how to calculate the terms a_n without assuming a value for π.

First, however, we describe how to estimate the area of the disc using *outer* polygons. These give an upper estimate for π. Once again we start with a regular hexagon and repeatedly double the number of sides. The method of calculating the areas of these polygons is illustrated in Figure 24 and the explanation is given below.



$$\text{area} = 6\tan(\pi/6) = 6 \times 1/\sqrt{3} \approx 3.464$$

$$\text{area} = 12\tan(\pi/12) = 12(2-\sqrt{3}) \approx 3.215$$

Figure 24 The area of regular outer polygons

Let s_n denote the number of sides of the nth outer polygon. As before, $s_n = 3 \times 2^n$, for $n = 1, 2, \ldots$. Also let b_n denote the area of the nth outer polygon. This nth outer polygon consists of s_n isosceles triangles, each of height 1 and base $2\tan(\pi/s_n)$. Thus

$$b_n = s_n \tan(\pi/s_n), \quad n = 1, 2, \ldots. \tag{13}$$

For example (to 3 d.p.) we have

$$b_1 = 3.464, \quad b_2 = 3.215, \quad \ldots, \quad b_6 = 3.142.$$

Geometrically, it is clear that each time we double the number of sides of the outer polygon, the area decreases, so

$$b_1 > b_2 > b_3 > \cdots > b_n > b_{n+1} > \cdots.$$

Hence the sequence (b_n) is (strictly) decreasing and bounded below (by 0, for example). Thus, by the Monotone Convergence Theorem, (b_n) is also convergent. Intuitively, we expect that (b_n) has the same limit as (a_n), which we have defined to be π. But how can we prove this?

It is a remarkable fact that the terms a_n and b_n can be calculated by using the following equations, known as the *Archimedean algorithm*:

$$a_{n+1} = \sqrt{a_n b_n}, \quad n = 1, 2, \ldots, \tag{14}$$

$$b_{n+1} = \frac{2a_{n+1}b_n}{a_{n+1} + b_n}, \quad n = 1, 2, \ldots. \tag{15}$$

These equations for calculating a_n and b_n can be deduced from equations (12) and (13) by using trigonometric identities, though we do not prove this here.

Starting with $a_1 = \frac{3}{2}\sqrt{3} = 2.598\ldots$ and $b_1 = 2\sqrt{3} = 3.464\ldots$, we can use these equations iteratively to calculate first $a_2 = \sqrt{a_1 b_1}$, then $b_2 = 2a_2 b_1/(a_2 + b_1)$, and so on. Here are the first few values (to three decimal places) of each sequence obtained in this way.

s_n	6	12	24	48	96	192
a_n	2.598	3	3.106	3.133	3.139	3.141
b_n	3.464	3.215	3.160	3.146	3.143	3.142

It appears that the sequence (b_n) converges to the same limit as (a_n). Indeed, by equation (14), we have $b_n = a_{n+1}^2/a_n$, so

$$\lim_{n\to\infty} b_n = \frac{\left(\lim\limits_{n\to\infty} a_{n+1}\right)^2}{\lim\limits_{n\to\infty} a_n} = \frac{\pi^2}{\pi} = \pi,$$

by the Combination Rules and our definition of π.

However, the convergence of these sequences (a_n) and (b_n) to $\pi = 3.141\,59\ldots$ seems quite slow. In Unit F4 *Power series* we give other ways to calculate π, which are more efficient, and we show that π is an irrational number. All these methods use the Monotone Convergence Theorem in some way.

5.3 The number e

You will have seen that the number e plays an important role in mathematics. There are various ways in which e is defined, one of which is as the limit of the sequence (a_n) where

$$a_n = \left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, \ldots.$$

Thomas Harriot

Early in the 17th century, the English mathematician Thomas Harriot (1560–1621), while working on the problem of compound interest, recognised that the sequence $a_n = (1 + 1/n)^n$ had a limit but did not give it a value. Since he did not publish his work, his results remained unknown until the much later study of his manuscripts. The first attempt to find a value for the limit of the sequence was in 1683 by Jacob Bernoulli (1654–1705), who was also working on the problem of compound interest and who calculated the limit to lie between 2 and 3. In 1748 Leonhard Euler, in his *Analysis Infinitorum*, showed that the limit is e and he calculated its approximate value to 18 decimal places.

In this subsection we use the Monotone Convergence Theorem to prove that the limit of the sequence (a_n) exists. To do this, we prove that the sequence (a_n) is increasing and bounded above. If we plot the first few terms on a sequence diagram, then it certainly seems that these properties hold; see Figure 25.

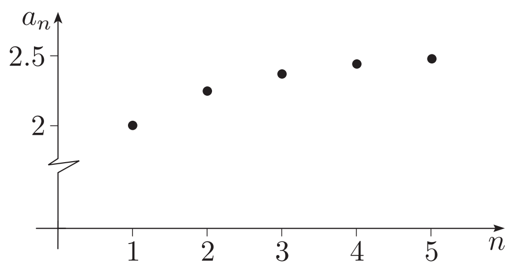


Figure 25 The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

We prove these facts by using the Binomial Theorem:

$$a_n = \left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^n.$$

As n increases, the number of terms in this sum increases and the new terms are all positive. Also, for each fixed $k \geq 1$ and any $n \geq k$, the $(k+1)$th term of the sum is

$$\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k = \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right),$$

and the product on the right increases as n increases (because each of the factors does). Hence the sequence (a_n) is increasing.

To see that this sequence is bounded above, note that the $(k+1)$th term of the above sum satisfies the inequality

$$\frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \leq \frac{1}{k!},$$

since each of the expressions in brackets is at most 1. Hence

$$a_n = \left(1+\frac{1}{n}\right)^n \leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$
$$\leq 1+1+\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}},$$

since $k! = k(k-1)\times\cdots\times 2\times 1 \geq 2^{k-1}$, for $k = 1, 2, \ldots$.

Now we use the fact that the the sum of a finite geometric series is given by

$$1+r+\cdots+r^{n-1} = \frac{1-r^n}{1-r}$$

and so, in the case $r = \frac{1}{2}$, we have

$$1+\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 2-\frac{1}{2^{n-1}}.$$

Thus

$$a_n = \left(1+\frac{1}{n}\right)^n \leq 3-\frac{1}{2^{n-1}}, \quad \text{for } n = 1, 2, \ldots$$

so (a_n) is bounded above by 3.

Hence, by the Monotone Convergence Theorem, the sequence (a_n) is convergent, with limit at most 3. This allows us to make the following definition.

Definition

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

For larger and larger values of n, the terms $a_n = (1+1/n)^n$ give better and better approximate values for e. However, the sequence (a_n) converges to e rather slowly, and we need to take very large integers n to get a reasonable approximation to $e = 2.718\,28\ldots$. For example,

$$\left(1+\tfrac{1}{1000}\right)^{1000} = 2.716\ldots.$$

In Unit D3 *Series* we give another way to calculate e, which is more efficient, and we show that e is an irrational number.

Similar arguments to the ones given here can be used to prove that

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,$$

for any $x > 0$. In fact, this is true for all real values of x but different arguments are needed when x is not positive.

Summary

In this unit you have studied sequences of real numbers denoted by (a_n), and looked at how the ideas of convergence and the limit of a sequence can be made precise. You have seen that a key role is played by null sequences, that is, sequences which converge to zero, and met a list of basic null sequences. You have learnt how to use these basic null sequences, together with the Combination Rules and the Squeeze Rule, to show that other sequences are convergent and to find their limits.

You have also seen that many sequences are divergent. These include sequences that tend to infinity or to minus infinity, which can be identified by using the Reciprocal Rule. More generally, you have seen that a sequence is divergent if it has a subsequence that tends to infinity or minus infinity, or if it has two subsequences which converge to different limits.

Finally, you met the Monotone Convergence Theorem, which states that any increasing sequence which is bounded above must be convergent. You have also seen this theorem applied to show that particular sequences converge to the numbers π and e.

Sequences play a key role in the remaining analysis units of this module, so it is important that you have a good understanding of the material in this unit.

Learning outcomes

After working through this unit, you should be able to:

- draw the *sequence diagram* of a given sequence
- explain what is meant by a *monotonic* sequence
- explain the meaning of the phrase 'a sequence *eventually* has a given property'
- explain the definition of *null sequence* and apply it in simple cases
- use the Power Rule, the Combination Rules and the Squeeze Rule to test for null sequences
- recognise certain *basic* null sequences
- explain what is meant by the terms *convergent sequence* and *limit of a sequence*, and by the statements $\lim\limits_{n\to\infty} a_n = l$, or $a_n \to l$ as $n \to \infty$
- use the Combination Rules to calculate limits of sequences
- state and use some theorems about convergent sequences
- explain the terms *divergent* sequence, *bounded* sequence and *unbounded* sequence
- explain the phrases (a_n) *tends to infinity* and (a_n) *tends to minus infinity*, and use the Reciprocal Rule to recognise sequences which tend to infinity

- use the Subsequence Rules to recognise divergent sequences
- state the Monotone Convergence Theorem
- understand the role of the Monotone Convergence Theorem in the definitions of the numbers π and e.

Solutions to exercises

Solution to Exercise D22

(a) $4, 7, 10, 13, 16.$

(b) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}.$

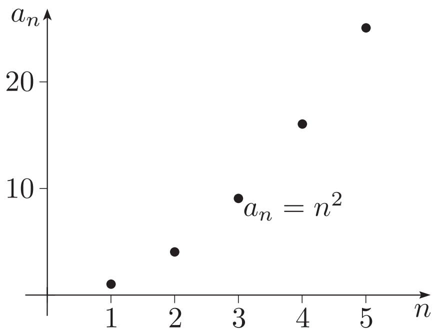
(c) $-1, 2, -3, 4, -5.$

(d) $1, 2, 6, 24, 120.$

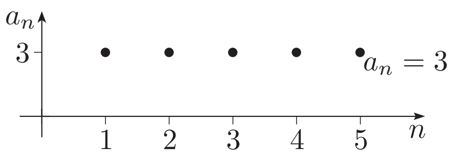
(e) $2, 2.25, 2.37, 2.44, 2.49.$

Solution to Exercise D23

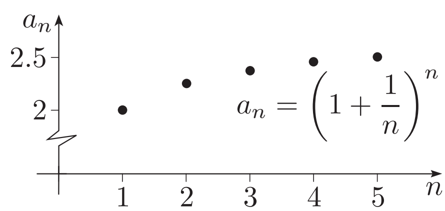
(a)



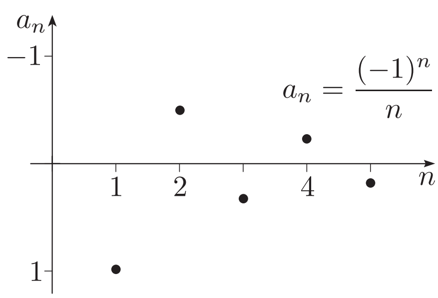
(b)



(c)



(d)



Solution to Exercise D24

(a) Since $a_n > 0$ for all n, we can use Strategy D4. We have

$$a_n = (n-1)! \quad \text{and} \quad a_{n+1} = n!,$$

so

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{n!}{(n-1)!} = \frac{n \times (n-1)!}{(n-1)!} \\ &= n \geq 1, \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

Thus (a_n) is increasing, so (a_n) is monotonic. (Notice, however, that (a_n) is not strictly increasing, since $a_1 = a_2 = 1$.)

(b) Since $a_n > 0$ for all n, we can use Strategy D4. We have

$$a_n = 2^{-n} \quad \text{and} \quad a_{n+1} = 2^{-(n+1)},$$

so

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{2^n}{2^{n+1}} = \frac{2^n}{2 \times 2^n} \\ &= \frac{1}{2} < 1, \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

Thus (a_n) is strictly decreasing, so (a_n) is monotonic.

(c) We use Strategy D3. We have

$$a_n = n + \frac{1}{n} \quad \text{and} \quad a_{n+1} = n + 1 + \frac{1}{n+1},$$

so

$$\begin{aligned}a_{n+1} - a_n &= \left(n + 1 + \frac{1}{n+1}\right) - \left(n + \frac{1}{n}\right) \\ &= 1 - \frac{1}{n(n+1)} > 0, \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

Thus (a_n) is strictly increasing, so (a_n) is monotonic.

Solution to Exercise D25

(a) True: $2^n > 1000$, for $n > 9$, since (2^n) is increasing and $2^{10} = 1024$.

(b) False: all the terms $a_1, a_3, a_5, \ldots$ are negative since

$$(-1)^n = -1, \text{ for } n = 1, 3, 5, \ldots.$$

(c) True: $\dfrac{1}{n} < 0.025$, for $n > \dfrac{1}{0.025} = 40$.

(d) True: $a_n > 0$ for all n, and

$$\frac{a_{n+1}}{a_n} = \frac{1}{4}\left(\frac{n+1}{n}\right)^4.$$

Using the rules for rearranging inequalities, we have

$$\begin{aligned}\frac{1}{4}\left(\frac{n+1}{n}\right)^4 \leq 1 \iff & \left(\frac{n+1}{n}\right)^4 \leq 4\\ \iff & 1+\frac{1}{n} \leq 4^{1/4}\\ \iff & \frac{1}{n} \leq \sqrt{2}-1\\ \iff & n \geq \frac{1}{\sqrt{2}-1} \approx 2.414.\end{aligned}$$

So

$$\frac{a_{n+1}}{a_n} \leq 1, \quad \text{for } n > 2.$$

Hence

$$a_{n+1} \leq a_n, \quad \text{for } n > 2,$$

so (a_n) is eventually decreasing.

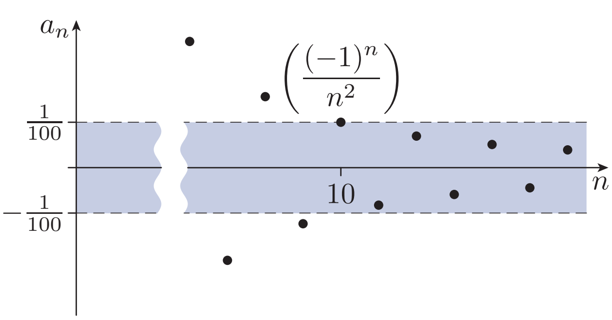
Solution to Exercise D26

Sequence diagrams are given here to aid your understanding, but you are not expected to have drawn these as part of your solutions.

(a) We have that

$$\begin{aligned}\left|\frac{(-1)^n}{n^2}\right| < \frac{1}{100} \iff & \frac{1}{n^2} < \frac{1}{100}\\ \iff & n^2 > 100\\ \iff & n > 10.\end{aligned}$$

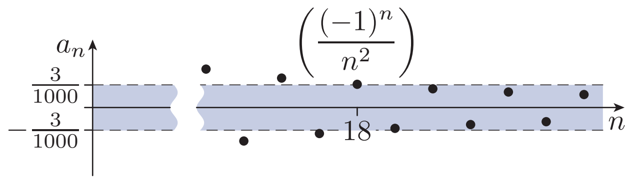
Hence we may take $N = 10$. This is illustrated below.



(b) We have that

$$\begin{aligned}\left|\frac{(-1)^n}{n^2}\right| < \frac{3}{1000} \iff & \frac{1}{n^2} < \frac{3}{1000}\\ \iff & n^2 > \frac{1000}{3}\\ \iff & n > \sqrt{\frac{1000}{3}} \approx 18.26.\end{aligned}$$

Hence we may take $N = 18$. This is illustrated below.



Solution to Exercise D27

(a) The sequence (a_n) is null. To prove this, we want to show that:

for each $\varepsilon > 0$, there is an integer N such that
$$\frac{1}{2n-1} < \varepsilon, \quad \text{for all } n > N. \qquad (*)$$

We know that

$$\begin{aligned}\frac{1}{2n-1} < \varepsilon \iff & 2n-1 > \frac{1}{\varepsilon}\\ \iff & n > \frac{1}{2}\left(1+\frac{1}{\varepsilon}\right),\end{aligned}$$

so statement $(*)$ holds if we take $N \geq \frac{1}{2}(1+1/\varepsilon)$.

Hence (a_n) is null.

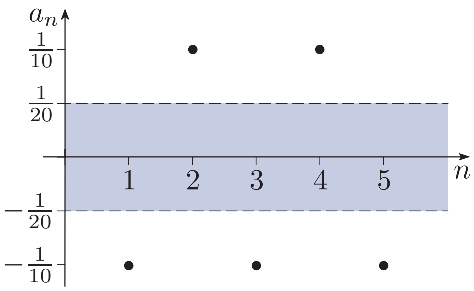
(b) The sequence (a_n) is not null. To prove this, we must find a positive value of ε for which there is no integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

Since

$$|a_n| = \left|\frac{(-1)^n}{10}\right| = \frac{1}{10}, \quad \text{for } n = 1, 2, \ldots,$$

we can take $\varepsilon = \frac{1}{20}$. This is illustrated in the following diagram. In this particular case, *all* the terms of the sequence lie outside the ε-strip for this value of ε.



(c) The sequence (a_n) is null. To prove this, we want to show that:

for each $\varepsilon > 0$, there is an integer N such that
$$\left|\frac{(-1)^n}{n^4+1}\right| < \varepsilon, \quad \text{for all } n > N. \qquad (*)$$

We know that
$$\left|\frac{(-1)^n}{n^4+1}\right| = \frac{1}{n^4+1}, \quad \text{for } n = 1, 2, \ldots,$$
and
$$\frac{1}{n^4+1} < \varepsilon \iff n^4 + 1 > \frac{1}{\varepsilon}$$
$$\iff n^4 > \frac{1}{\varepsilon} - 1.$$

Now $1/\varepsilon - 1$ is sometimes positive and sometimes negative, so we need to consider two cases.

If $\varepsilon \geq 1$, then $1/\varepsilon - 1 \leq 0$, so
$$n^4 > \frac{1}{\varepsilon} - 1, \quad \text{for } n = 1, 2, \ldots.$$

Hence statement $(*)$ holds with $N = 1$.

If $0 < \varepsilon < 1$, then $1/\varepsilon - 1 > 0$, so we can use Rule 5 for rearranging inequalities, giving
$$n^4 > \frac{1}{\varepsilon} - 1 \iff n > \left(\frac{1}{\varepsilon} - 1\right)^{1/4}.$$

Hence statement $(*)$ holds if we take $N \geq (1/\varepsilon - 1)^{1/4}$.

Thus statement $(*)$ holds in either case, so (a_n) is null.

Solution to Exercise D28

(a) We know that the sequence $\left(\frac{1}{2n-1}\right)$ is null, so (a_n) is null, by the Power Rule.

(b) We know that the sequence $\left(\frac{1}{n^3}\right)$ is null, so (a_n) is null, by the Multiple Rule.

(c) The sequences $\left(\frac{1}{n}\right)$ and $\left(\frac{1}{2n-1}\right)$ are null, so the sequences $\left(\frac{1}{n^4}\right)$ and $\left(\frac{1}{(2n-1)^{1/3}}\right)$ are also null, by the Power Rule.

Hence (a_n) is null, by the Product Rule and the Multiple Rule.

Solution to Exercise D29

(a) We guess that (a_n) is dominated by (b_n), where
$$b_n = \frac{1}{n}, \quad n = 1, 2, \ldots.$$

To check this, we have to show that
$$\frac{1}{n^2+n} \leq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

This holds because
$$n^2 + n \geq n, \quad \text{for } n = 1, 2, \ldots.$$

Since (b_n) is null, we deduce that (a_n) is null, by the Squeeze Rule.

(b) We guess that (a_n) is dominated by (b_n), where
$$b_n = \frac{1}{n}, \quad n = 1, 2, \ldots.$$

To check this, we have to show that
$$\left|\frac{(-1)^n}{n!}\right| \leq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

This holds because
$$\left|\frac{(-1)^n}{n!}\right| = \frac{1}{n!}$$
and
$$n! \geq n, \quad \text{for } n = 1, 2, \ldots.$$

Since (b_n) is null, we deduce that (a_n) is null, by the Squeeze Rule.

(c) We guess that (a_n) is dominated by (b_n), where
$$b_n = \frac{1}{n^2}, \quad n = 1, 2, \ldots.$$

To check this, we have to show that
$$\left|\frac{\sin(n^2)}{n^2+2^n}\right| \leq \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

This holds because $|\sin(n^2)| \leq 1$ and

$$n^2 + 2^n \geq n^2, \quad \text{for } n = 1, 2, \ldots.$$

Since (b_n) is null (by the Power Rule), we deduce, by the Squeeze Rule, that (a_n) is null.

Solution to Exercise D30

(a) (a_n) is a basic null series of type (b), with $c = 0.9$.

(b) (a_n) is a basic null series of type (d), with $c = 27$.

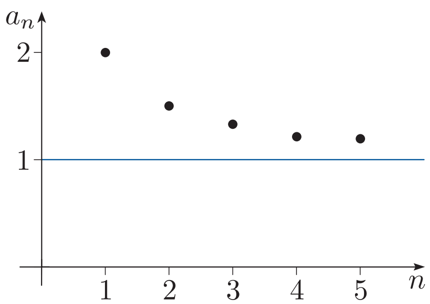
(c) (a_n) is a basic null series of type (a), with $p = \frac{1}{2}$.

(d) (a_n) is a basic null series of type (e), with $p = 27$.

(e) (a_n) is a basic null series of type (c), with $p = 1$ and $c = \frac{1}{2}$.

Solution to Exercise D31

(a)



The sequence diagram suggests that (a_n) converges to 1.

(b) $b_n = a_n - 1 = \dfrac{n+1}{n} - 1 = \dfrac{1}{n}$.

Hence (b_n) is a null sequence.

Solution to Exercise D32

We have

$$a_n - \frac{1}{2} = \frac{n^3+1}{2n^3} - \frac{1}{2} = \frac{1}{2n^3}.$$

We know that $\left(\dfrac{1}{2n^3}\right)$ is a null sequence, by the Multiple Rule, so (a_n) converges to $\frac{1}{2}$.

Solution to Exercise D33

In each case we apply Strategy D7.

(a) The dominant term is n^3, so we write

$$a_n = \frac{n^3 + 2n^2 + 3}{2n^3 + 1} = \frac{1 + 2/n + 3/n^3}{2 + 1/n^3}.$$

Since $(1/n)$ and $(1/n^3)$ are basic null sequences,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1 + 2/n + 3/n^3}{2 + 1/n^3} \\ &= \frac{1+0+0}{2+0} = \frac{1}{2},\end{aligned}$$

by the Combination Rules.

(b) The dominant term is 3^n, so we write

$$a_n = \frac{n^2 + 2^n}{3^n + n^3} = \frac{n^2/3^n + (2/3)^n}{1 + n^3/3^n}.$$

Since $n^2/3^n = n^2(1/3)^n$ and $n^3/3^n = n^3(1/3)^n$, we see that $(n^2/3^n)$, $((2/3)^n)$ and $(n^3/3^n)$ are all basic null sequences, so

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2/3^n + (2/3)^n}{1 + n^3/3^n} \\ &= \frac{0+0}{1+0} = 0,\end{aligned}$$

by the Combination Rules.

(c) The dominant term is $n!$, so we write

$$a_n = \frac{n! + (-1)^n}{2^n + 3n!} = \frac{1 + (-1)^n/n!}{2^n/n! + 3}.$$

Since $((-1)^n/n!)$ and $(2^n/n!)$ are basic null sequences,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1 + (-1)^n/n!}{2^n/n! + 3} \\ &= \frac{1+0}{0+3} = \frac{1}{3},\end{aligned}$$

by the Combination Rules.

Solution to Exercise D34

(a) By Rule 5 for rearranging inequalities with $p = n$, we have

$$n^{1/n} \le 1 + \sqrt{\frac{2}{n-1}} \iff n \le \left(1 + \sqrt{\frac{2}{n-1}}\right)^n.$$

Using the hint with $x = \sqrt{2/(n-1)}$, we obtain

$$\left(1 + \sqrt{\frac{2}{n-1}}\right)^n \ge \frac{n(n-1)}{2!}\left(\sqrt{\frac{2}{n-1}}\right)^2 = \frac{n(n-1)}{2}\,\frac{2}{n-1} = n.$$

Thus the right-hand inequality holds for $n \ge 2$, so it follows that the left-hand inequality also holds for $n \ge 2$, as required.

(b) For $n \ge 1$, we have $n^{1/n} \ge 1$. Combining this inequality with that in part (a), we obtain

$$1 \le n^{1/n} \le 1 + \sqrt{\frac{2}{n-1}}, \quad \text{for } n \ge 2.$$

Now the sequence (b_n) defined by

$$b_n = \sqrt{\frac{2}{n-1}}, \quad n = 2, 3, \ldots$$

is the same as the sequence defined by

$$b_n = \sqrt{\frac{2}{n}}, \quad n = 1, 2, \ldots.$$

So, since $(1/n)$ is a basic null sequence, it follows from the Multiple Rule and the Power Rule that (b_n) is null. Thus, by the Combination Rules,

$$\lim_{n\to\infty}\left(1 + \sqrt{\frac{2}{n-1}}\right) = \lim_{n\to\infty}(1 + b_n) = 1,$$

so, by the Squeeze Rule,

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Solution to Exercise D35

(a) This sequence is bounded because $1 + (-1)^n$ takes only the values 0 and 2, so

$$|1 + (-1)^n| \le 2, \quad \text{for } n = 1, 2, \ldots.$$

(b) This sequence is unbounded. Given any positive number M, there is a positive integer n such that $|(-1)^n n| = n > M$.

(c) This sequence is bounded because

$$\left|\frac{2n+1}{n}\right| = 2 + \frac{1}{n} \le 3, \quad \text{for } n = 1, 2, \ldots.$$

Solution to Exercise D36

(a) This sequence is unbounded and hence divergent, by Corollary D15.

(b) This sequence is convergent (with limit 1) and hence bounded, by Theorem D14. In fact,

$$a_n = \frac{n^2 + n}{n^2 + 1} \le \frac{n^2 + n^2}{n^2} = 2, \quad \text{for } n = 1, 2, \ldots.$$

(c) This sequence is unbounded and hence divergent, by Corollary D15.

(d) The first few terms of this sequence are

$$1, 2, \tfrac{1}{3}, 4, \tfrac{1}{5}, 6, \ldots.$$

This sequence is unbounded because, given any positive number M, there is an even integer $2n$ such that $(2n)^{(-1)^{2n}} = 2n > M$. Hence the sequence is divergent, by Corollary D15.

Solution to Exercise D37

(a) Each term of (a_n) is positive and

$$\frac{1}{a_n} = \frac{n}{2^n}$$

is a basic null sequence. Hence $a_n \to \infty$, by the Reciprocal Rule.

(b) The dominant term is 2^n, so we first write

$$2^n - n^9 = 2^n\left(1 - \frac{n^9}{2^n}\right), \quad \text{for } n = 1, 2, \ldots.$$

Since $(n^9/2^n)$ is a basic null sequence, it follows that $(n^9/2^n)$ is eventually less than 1, so (a_n) is eventually positive.

Next we write

$$\frac{1}{a_n} = \frac{1}{2^n - n^9} = \frac{1/2^n}{1 - n^9/2^n}.$$

Now $(1/2^n)$ and $(n^9/2^n)$ are basic null sequences, so, by the Combination Rules,

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0}{1 - 0} = 0.$$

Hence $a_n \to \infty$, by the Reciprocal Rule.

(c) We know that $2^n/n \to \infty$, by part (a), and

$$a_n = 2^n/n + 5n^9 \geq 2^n/n, \quad \text{for } n = 1, 2, \ldots.$$

Hence $a_n \to \infty$, by the Squeeze Rule for sequences which tend to infinity.

(Alternatively, you could have used the Reciprocal Rule or the Sum and the Multiple Rules.)

(d) Each term of (a_n) is positive. The dominant term is 2^n, so we write

$$\frac{1}{a_n} = \frac{n^9 + n}{2^n + n^2} = \frac{n^9/2^n + n/2^n}{1 + n^2/2^n}.$$

Now $(n/2^n)$, $(n^2/2^n)$ and $(n^9/2^n)$ are basic null sequences so, by the Combination Rules,

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0+0}{1+0} = 0.$$

Hence $a_n \to \infty$, by the Reciprocal Rule.

Solution to Exercise D38

(a) (i) $a_2 = 4$, $a_4 = 16$, $a_6 = 36$, $a_8 = 64$, $a_{10} = 100$.

(ii) $a_3 = 9$, $a_7 = 49$, $a_{11} = 121$, $a_{15} = 225$, $a_{19} = 361$.

(iii) $a_1 = 1$, $a_4 = 16$, $a_9 = 81$, $a_{16} = 256$, $a_{25} = 625$.

(b) The first three terms of the odd subsequence are $a_1 = 1$, $a_3 = \frac{1}{3}$, $a_5 = \frac{1}{5}$; the first three terms of the even subsequence are $a_2 = 2$, $a_4 = 4$, $a_6 = 6$.

Solution to Exercise D39

(a) We have

$$a_{2k} = 1 + \frac{1}{2k} \quad \text{and} \quad a_{2k-1} = -1 + \frac{1}{2k-1},$$

for $k = 1, 2, \ldots$. Thus

$$\lim_{k\to\infty} a_{2k} = 1, \quad \text{whereas} \quad \lim_{k\to\infty} a_{2k-1} = -1.$$

Hence (a_n) is divergent, by the First Subsequence Rule.

(b) We have

$$a_{3k} = k - \lfloor k \rfloor = 0$$

and

$$a_{3k+1} = k + \tfrac{1}{3} - \lfloor k + \tfrac{1}{3} \rfloor = k + \tfrac{1}{3} - k = \tfrac{1}{3},$$

for $k = 1, 2, \ldots$. Thus

$$\lim_{k\to\infty} a_{3k} = 0, \quad \text{whereas} \quad \lim_{k\to\infty} a_{3k+1} = \tfrac{1}{3}.$$

Hence (a_n) is divergent, by the First Subsequence Rule.

(c) We have

$$a_1 = 1,\ a_2 = 0,\ a_3 = -3,$$

$$a_4 = 0,\ a_5 = 5,\ a_6 = 0, \ldots.$$

Now

$$\begin{aligned} a_{4k+1} &= (4k+1)\sin\left(2k\pi + \tfrac{1}{2}\pi\right) \\ &= 4k+1, \end{aligned}$$

for $k = 1, 2, \ldots$. Thus $a_{4k+1} \to \infty$ as $k \to \infty$.

Hence (a_n) is divergent, by the Second Subsequence Rule.

Solution to Exercise D40

Since (a_n) is increasing, it follows from the Monotonic Sequence Theorem that either (a_n) is convergent or $a_n \to \infty$.

If $a_n \to \infty$ as $n \to \infty$, then it follows from Theorem D19(b) that $a_{n_k} \to \infty$ as $k \to \infty$, and we know that this is false.

Hence (a_n) must be convergent.